Photon echo techniques in quantum information

Outline and suggested reading

Jevon Longdell, University of Otago

July 22, 2014

1. The Bloch sphere and the dynamics of driven two level atoms. I will assume that everyone is pretty familiar with this, and probably cover it in a single slide. Allen and Eberly \[1\] is a good reference. Sections 2.1–2.4, 3.1–3.3 and 9.1–9.4.

2. The Maxwell-Bloch equations. Combing the Optical bloch equations with Maxwells equations for a description of pulses travelling in though crystals. This is covered in the first part of Chapter 4 of Allen and Eberly

3. Photon echoes for quantum memories without rephasing pulses. The use of controlled reversible inhomogeneous broadening (CRIB) \[2, 3, 4\] and atomic frequency combs (AFC) for making echoes \[5\].

4. The quantised Maxwell-Bloch Equations and Input-Output theory. Here we replace the classical light fields of the Maxwell Bloch equations with the quantised light fields. And discuss when you do and when you don’t need the quantum treatment. Each thin slice of crystal now has a quantum mechanical input and an output field as introduced Collet and Gardiner \[6\] \[7\]. The first time we used this formalism was in the RASE paper \[8\].

5. The noise in the two pulse echo and rephased amplified spontaneous emission Why the two pulse photon echo is too noisy to be a quantum memory, and how you can use this noise for entanglement generation. \[9\] \[8\]

6. Photon echoes for quantum memories with rephasing pulses. Ways to make optically rephased photon echoes quiet. The hybrid photot echo rephasing (HYPER) \[10\] and revival of a silenced echo (ROSE) techniques.

7. Photon echoes and resonators cavities. Cavity enhancement of the echo based memory and entanglement generation \[11\] \[12\] \[13\] \[14\]

8. More fun with rare earths and resonators. Single atoms, and bad cavity cQED \[15\]. Microwave upconversion \[16\] \[17\].

The references with the * are suggested reading: \[3, 6, 8, 11, 14\].

References


Electro-Optic Quantum Memory for Light Using Two-Level Atoms

G. Héret,1 J. J. Longdell,2,3 A. L. Alexander,2 P. K. Lam,1 and M. J. Sellars2,*

1ARC COE for Quantum-Atom Optics, Australian National University, Canberra, ACT 0200, Australia
2Laser Physics Centre, RSPhysSE, Australian National University, Canberra, ACT 0200, Australia
3Department of Physics, University of Otago, Dunedin, New Zealand

(Received 12 March 2007; published 16 January 2008)

We present a simple quantum memory scheme that allows for the storage of a light field in an ensemble of two-level atoms. The technique is analogous to the NMR gradient echo for which the imprinting and recalling of the input field are performed by controlling a linearly varying broadening. Our protocol is perfectly efficient in the limit of high optical depths and the output pulse is emitted in the forward direction. We provide a numerical analysis of the protocol together with an experiment performed in a solid state system. In close agreement with our model, the experiment shows a total efficiency of up to 15%, and a recall efficiency of 26%. We suggest simple realizable improvements for the experiment to surpass the no-cloning limit.

DOI: 10.1103/PhysRevLett.100.023601 PACS numbers: 42.50.Gy, 42.50.Md

Some of the most significant advances in quantum information processing have been made using quantum optics. To extend these techniques, it is necessary to have devices such as single photon sources and quantum repeaters using quantum memories, where information is exchanged in a controlled fashion between light fields and material systems. A quantum memory for light is a device that can efficiently delay or store the quantum states of light fields. This is usually achieved via some form of imprinting onto an atomic system. The quantum states stored must also be faithfully retrievable on demand and the total efficiency of the processes must exceed the classical benchmark so that quantum information can be retained [1]. It has been proposed that the requisite control and strong coupling can both be achieved using an ensemble approach, where the light field interacts with a large number of identical atoms [2]. Classical pulses [3] and single photon states [4] have been stored and retrieved using electromagnetically induced transparency from warm vapor cells and magneto-optical traps. In the continuous variable regime, quantum states of light have been mapped onto atoms using the off resonant interaction of light with spin polarized cesium vapors [5]. Further improvements are nevertheless still needed for realizing an efficient and reliable quantum memory.

In 2001, Moiseev and Kroell [6] published a proposal for a quantum memory for light based on modified photon echoes. In contrast to a normal photon echo [7–9], the rephasing came from controlled reversible inhomogeneous broadening (CRIB). This proposal has then been generalized to other broadening mechanisms [10–12]. Storage of multiple pulses using CRIB has been demonstrated using Stark shifts in europium dopants [13,14] and work towards demonstrating such echoes in other systems has been carried out [11,15]. The quantum memory proposals using CRIB operate via a time reversal of the storage process. It has been shown that by reversing the detunings of the atoms, the equations of motion for light traveling in the backward direction describe the motion of a time reversed copy traveling in the forward direction [10]. The pulse can exit the ensemble in the backward direction by applying a phase matching operation to the atoms with an auxiliary third atomic level and π pulses. Sangouard et al. [16] have recently shown that a quantum memory can be achieved without any external fields. However, using this approach, the efficiency would be limited to a maximum of 54%.

In this Letter, we show that 100% efficiency using CRIB is possible using only two level atoms Stark shifted by an external electric field. The only light seen by the atomic ensemble during the entire process is then the light field of interest and the echo propagates in the forward direction without being reabsorbed provided the Stark shift is linear along the sample length. The principle benefit of such a two-level scheme lies in its simplicity. First, the absence of phase matching π pulses greatly simplifies the implementation. The precision of the electro-optic switching is not as critical as the π pulse parameters. The scheme is also more robust than those using optic-optic interactions, where cross-coupling and transverse-modal effects on the beams may reduce the process efficiency. Second, as the memory requires only two atomic levels, this scheme is applicable to many more atomic systems. In particular, erbium dopants which allow operation at the telecommunication wavelength of 1.5 μm have been shown to have very good two-level characteristics [17], while a lambda system has yet to be demonstrated.

To demonstrate the efficiency of the memory, we consider the interaction between a collection of N two-level atoms and a quantum optical field with slowly varying envelope $\hat{E}(z, t)$. As shown in Fig. 1(a), the pulse with duration $t_{\text{pulse}}$ enters the medium at $z = -z_0$, $t = -t_0$ and the detuning of the atoms are flipped at $t = 0$. We follow the same procedure as in Ref. [18] and use locally averaged atomic operators. The Heisenberg-Langevin
equations describing the interaction in a moving frame at the speed of light are

\[ \alpha = \left( \gamma / 2 + i \eta z \right) \alpha + i g \hat{E} (\hat{\sigma}_g - \hat{\sigma}_e) + \hat{F}_\alpha, \]  

(1)

\[ \dot{\sigma}_g = \gamma \hat{\sigma}_e + i g (\hat{E}^\dagger \alpha - \hat{E} \alpha^\dagger) + \hat{F}_g, \]  

(2)

\[ \partial_z \hat{E} = igN \alpha, \]  

(3)

where \( \alpha \) is the atomic polarization operator, \( \sigma_{e/g} \) the population of the excited or ground state, \( g \) the atomic transition coupling strength, and \( i \eta z \) is the linearly varying detuning from resonance. We also introduced \( \gamma \) as a decay rate from the excited state and the corresponding Langevin operators. Under a small pulse approximation, ensuring that a negligible amount of atoms reaches the excited state (\( \sigma_e \ll \sigma_g \)), the nonzero noise correlations are found to be [19]

\[ \langle \hat{F}_\alpha (z_1, t_1) \hat{F}_\alpha^\dagger (z_2, t_2) \rangle = 2\gamma \frac{\delta (z_1 - z_2) \delta (t_1 - t_2)}{n \mathcal{A}}, \]  

(4)

where \( \mathcal{A} \) is the cross section area of the beam and \( n \) the atomic density. Because the spontaneous excitation of atoms to the excited states requires large energy, the noise arises from the normally ordered Langevin correlations and is therefore vacuum noise [19]. Furthermore, the Heisenberg-Langevin equations are linear after the weak probe approximation so the atomic and optical field variables can be treated as \( c \)-numbers [18]. These two results ensure that transmissivity is the only quantity needed to fully characterize the memory.

The effective optical depth of the sample is \( gN/\eta \). This ratio indeed quantifies the portion of the input light that is not stored in the sample, a feature of the absorption of short pulses by narrow linewidth atoms [20]. \( 2\eta \zeta_0 \) on the other hand gives the spectral coverage of the absorption process. We will here let \( gN/\eta = 10/3 \) and \( 2\eta \zeta_0 = 2/\tau_{\text{pulse}} \) to ensure an optimum writing efficiency. The sign of the Stark shift will be reversed after the complete absorption of the pulse. Figures 1(b) and 1(c) show the results of numerical simulations with a zero decay rate \( \gamma = 0 \). Figure 1(b) reveals that with these parameters, a large echo comes out of the sample after flipping the electric field. The excitation in the medium can be thought of as a polariton asymptotically slowing down to zero velocity with the asymptotic limit for the different spectral components of the pulse distributed spatially along the propagation direction. When the sign of the Stark shift is reversed, the polariton accelerates in the forward direction out of the sample. Because of the monotonicity of the Stark shift with space, all the spectral components of the pulse escape the sample without being resonant with the atoms. A detailed description of this light-matter superposition in this scheme will be presented elsewhere.

The importance of the Stark-shift monotonicity is highlighted further in Figs. 2(a) and 2(b). The figures show plots of the electro-optic memory efficiencies as a function of optical depth. It can be seen in Fig. 2(a) that the efficiency of our memory asymptotes to 100% with optical depth. It was shown in Ref. [16] however, that the efficiency of such a two-level atom electro-optic memory is limited when the broadening is nonmonotonic. To allow a direct comparison between the two-level atom technique analyzed in Ref. [16] and our scheme, we show in Fig. 2(b)
the result of simulations using a nonmonotonic broadening. A complete agreement with the results of Ref. [16] is found. The efficiency reaches a maximum of 54% and at high optical depths, the optical information is retained in the medium. Our modeling also demonstrates that a small spatial nonlinearity of the Stark shift, $\eta_z$, and finite switching speed of the applied electric field have very little influence on the efficiency. As the efficiency of our electro-optic memory can be well above 50% with no excess noise, the echo is guaranteed to be the best possible copy of the input state [21], demonstrating that our scheme is a quantum memory for light.

Figure 3 shows a contour map of the real part of the electric field. Because of the large phase shift seen by the field when it enters the medium, the last atoms in the sample absorb the field a long time after the first atoms. The storage time is then required to be large enough for the atoms to reradiate the whole input pulse. Figure 3(a) presents simulations where the storage time is 4 times the pulse duration. In that situation, a time varying phase shift is present across the output pulse, so that it is frequency shifted with respect to the input. Figure 3(b) shows simulations where the Stark shift is flipped after nearly all the atoms have absorbed the field. The frequency shift is then eliminated and only a constant phase shift is present. One way of compensating for the phase shift would be to cascade two electro-optic memories using opposite switching procedures. Another method would be to use an electro-optic phase shifter driven with the appropriate voltage waveform.

The initial demonstrations of photon echoes via CRIB used a linear Stark shift [13], as in the current proposal. At the time it was thought that a more difficult experiment involving an auxiliary atomic level and counter propagating $\pi$ pulses would be required to achieve a quantum memory for light. The above analysis shows that this is not the case. Here we report enhancements of the efficiency by more than 5 orders of magnitudes compared with the initial Stark-echo demonstrations of Ref. [13] in complete agreement with the theory. A large part of this improvement is due to a change of the dopant ions used to praseodymium allowing larger optical depths to be reached.

The experiment was carried out on a spectral antihole which was prepared as described in Fig. 4. Light from a highly stabilized dye laser was frequency shifted and gated with acousto-optic modulators. The pulse was then steered toward the sample of Pr$^{3+}$:Y$_2$SiO$_5$ (0.05%). The sample was approximately a 4 mm cube and was held at temperatures in the range 2–4 K. Four electrodes were placed around the sample in a quadrupole arrangement and provided an electric field that varied linearly along the optical path. The electrodes were 1.7 mm diameter rods separated by 8 mm. Voltages of approximately ±5 V were used to broaden the antihole and were able to be switched in 1 $\mu$s. Heterodyne detection was used to detect the transmitted pulses. The beam diameter in the sample was approximately 200 $\mu$m and the corresponding pulse areas $\pi/20$.

Figure 5 shows the experimental traces of the electro-optic echo memory with and without the preparation of the two-level antihole. In the results, 49% of the incident light was transmitted straight through the sample (with less than 2% absorption without the antihole) and 15% of the total input light was recalled as an echo. The recall efficiency, defined as the ratio between the reemitted and absorbed

![FIG. 3 (color online). Real part of the optical field in a moving frame at c. At $t = -t_0$, the light field enters the sample and is gradually absorbed by the medium. At $t = 0$, the quadrupole field is flipped and the time reverse process commences producing a forward propagating pulse. For the parameters given in Fig. 1, with a storage time of $8\tau_{\text{pulse}}$, (a) shows a small phase shift across the retrieved pulse. With a storage time $80\tau_{\text{pulse}}$, (b) shows a near ideal pulse retrieval. The arrows denote the wave front of the light fields.](image)

![FIG. 4. Energy level diagram and spectral scheme for the praseodymium dopants in yttrium orthosilicate Pr$^{3+}$:Y$_2$SiO$_5$ (0.05%). The light at frequency $A$ is the light stored by the electro-optic memory. The natural inhomogeneous linewidth of the sample, $\omega_{\text{nat}}$, is a few GHz wide. To set up the experiment, the applied light is swept around frequency $A$ to create a spectral hole a few MHz wide, $\omega_{\text{hole}}$. Light at frequencies $B$ and $C$ is then applied to prepare a narrow antihole around $A$ with linewidth $\omega_{\text{anti}}$. Although the diagram shows the use of the $\pm 1/2$ excited states, there are ions contributing to the antihole from the $\pm 3/2$ and $\pm 5/2$ excited states due to inhomogeneous broadening in the optical transition. In our experiment $\omega_{\text{anti hole}} = 30$ kHz.](image)
light is 26%, showing the potential improvements of our system with larger optical depths. A minimal output pulse distortion was observed, which demonstrates the rather large time-bandwidth product in our experiment. Using our numerical model, we vary the spectral width of the unBroadened antihole and the optical depth of the sample to match the experimental results. Close agreement between the experimental results and the simulations is obtained only with these two free parameters. Our numerical model suggests an antihole width of 30 kHz, in agreement with the experimental expectation if the hyperfine transition broadening were the main limitation to the antihole width.

For a given pulse length, the optimization of the experiment is dependent on a compromise between increasing the ratio between applied and intrinsic broadenings and maximizing the optical depth of the sample. In our experiment, $2\zeta_0/\gamma$ is around 12 while the optical depth $gN/\eta = 0.06$. For each crystallographic site where praseodymium is located, there is another related to it by inversion. In order to implement a completely efficient memory, only one of the site pair can be used. In principle, this could be achieved by Stark shifting with a homogeneous electric field and optical pumping. In our experiment, however, both orientations were used. The theoretical modeling on Fig. 5 takes into account these two orientations by having two Bloch equations and two source terms for the optical field. Simulations suggest that using Fourier limited pulses, selecting only one orientation of the praseodymium ions, and increasing the optical depth by a factor of 3 would enable the scheme to reach more than 50% efficiency.

In conclusion, we have proposed an electro-optic quantum memory for light with a linear Stark shift. In contrast to existing quantum memories based on controlled inhomogeneous broadening, our scheme requires only two atomic levels and is therefore applicable to a wide range of systems. Moreover, our scheme does not require auxiliary optical pulses for the imprint and recall process. Our experiments show an efficiency of 15% and a time-bandwidth product of around three which compares favorably with the performance of quantum memories based on EIT [4]. The experiment is well modeled by the Maxwell-Bloch equations. Modest improvements on the experimental parameters will allow efficiencies higher than 50% to be achieved.

We thank C. Simon and B.C. Buchler for useful discussion. We acknowledge financial assistance from the Australian Research Council.

*matthew.sellars@anu.edu.au*

Squeezing of intracavity and traveling-wave light fields produced in parametric amplification

M. J. Collett and C. W. Gardiner

Physics Department, University of Waikato, Hamilton, New Zealand
(Received 6 February 1984)

A general input-output theory for quantum dissipative systems is developed in which it is possible to relate output to input via the internal dynamics of a system. This is applied to the problem of computing the squeezing produced by a degenerate parametric amplifier located inside a cavity. The results for the internal modes are identical with those of Milburn and Walls [Opt. Commun. 39, 401 (1981)]. The output field is also found to have only 50% of maximal squeezing. However, by taking the output for a degenerate parametric amplifier inside a single-ended cavity and feeding this into an empty single-ended cavity, one can produce a maximally squeezed state inside this second cavity.

I. INTRODUCTION

Recent calculations by Milburn and Walls \(^1\) and by Yurke \(^2\) have shown that squeezing in a parametric amplifier is a subject of great subtlety and possible ambiguity. Milburn and Walls made calculations using master-equation techniques which give a maximum squeezing of only \(\frac{1}{3}\) compared with a theoretical maximum of \(\frac{1}{2}\), while Yurke has carried out a single-mode analysis which appears to show that the actual output from the cavity is not so limited.

We show that a more careful formulation of input, output, and internal fields in such a system is needed. The behavior of the light field inside a cavity can be described by standard master-equation techniques, which treat the external field only in its role as a heat bath. This approach is incomplete in two ways: first, it does not allow for the possibility that the incoming part of the field may be other than a vacuum or thermal, although the inclusion of a classical driving field is equivalent to allowing the incoming field a coherent amplitude; second, and more importantly, it contains no prescription for calculating the properties of the light emitted from the cavity, despite the fact that it is precisely this emitted light which is normally accessible to measurement. The approach presented here aims to provide such a method. The internal field is linked with the input by identification of the "noise" with the incoming field, and the output can then be calculated using the boundary conditions at the cavity mirror. Yurke and Denker \(^3\) have treated the case of an electronic circuit connected to a transmission line from this viewpoint. To the circuit this looks just the same as a resistor, but it is clearly capable of carrying signals in and out.

II. INPUT-OUTPUT EQUATIONS FOR A MODEL CAVITY

We present here a phenomenological derivation of the input-output theory for a light field interacting with a cavity. In a later paper we will present a rigorous development, which is, however, not so intuitively appealing or instructive.

An optical cavity is commonly described by a Hamiltonian of the form

\[
H_{\text{tot}} = H_{\text{sys}} + H_b + H_{\text{int}},
\]

where \(H_{\text{sys}}\) is a function of internal-mode operators only, \(H_b\) is the free Hamiltonian of the bath, and \(H_{\text{int}}\) describes the interaction between bath and cavity field, which is taken to be linear. The behavior of the internal mode or modes may then be calculated by master-equation methods. Alternatively, one may obtain quantum Langevin equations \(^4\) which for a single-mode cavity becomes

\[
\frac{da}{dt} = - \frac{i}{\hbar} [a, H_{\text{sys}}] - \frac{\gamma}{2} a + \Gamma,
\]

where \(a\) is the annihilation operator for the internal mode, \(\gamma\) is the cavity damping constant, and \(\Gamma\) is the noise operator. For a single-ended cavity, the bath is simply the radiation field outside the mirror and the inhomogeneity \(\Gamma\) must therefore be ascribed to the incoming part of this external field

\[
\Gamma = \gamma a_{\text{in}},
\]

where \(a_{\text{in}}\) describes the incoming field and \(\gamma\) is as yet undetermined. Time reversal of (2) must be equivalent to a change of sign in the systematic part, and replacement of the incoming field by the outgoing one, to give

\[
\frac{da}{dt} = \frac{i}{\hbar} [a, H_{\text{sys}}] - \frac{\gamma}{2} a + \gamma a_{\text{out}}.
\]

There will be a boundary condition at the mirror which will take to be of the form

\[
a = k (a_{\text{in}} + a_{\text{out}})
\]

and consistency in (2)–(5) then requires \(\gamma = k \gamma\), giving

\[
\frac{da}{dt} = - \frac{i}{\hbar} [a, H_{\text{sys}}] - \frac{\gamma}{2} a + k \gamma a_{\text{in}}
\]

\[
= - \frac{i}{\hbar} [a, H_{\text{sys}}] + \frac{\gamma}{2} a - k \gamma a_{\text{out}}.
\]

For a linear system this can be rewritten as
\[
\frac{da}{dt} = \left( A - \frac{\gamma}{2} \right) a + k\gamma a_{\text{in}} \\
= \left( A + \frac{\gamma}{2} \right) a - k\gamma a_{\text{out}} ,
\]

(7)

where

\[
a = \begin{pmatrix} a \\ a^\dagger \end{pmatrix}
\]

(8)

and \( A \) is a matrix. This system is linear in a quite general sense, including, for instance, the possibility of phase conjugation (on phase-conjugating and phase-preserving amplifiers, see Caves\(^5\)). In terms of frequency components, defined by

\[
\bar{a}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} a(t) dt ,
\]

(9)

Eq. (7) becomes

\[
-i\bar{a}(\omega) = (A - \frac{\gamma}{2} \mathbb{1}) \bar{a}(\omega) + k\gamma \bar{a}_{\text{in}}(\omega) \\
= (A + \frac{\gamma}{2} \mathbb{1}) \bar{a}(\omega) - k\gamma \bar{a}_{\text{out}}(\omega) ,
\]

(10)

where

\[
\bar{a}(\omega) = \begin{pmatrix} \bar{a}(\omega) \\ \bar{a}^\dagger(-\omega) \end{pmatrix} ,
\]

(11)

and for simplicity, the commutation relations of the operators will be taken as

\[
\left[ \bar{a}_{\text{in}}(\omega), \bar{a}_{\text{in}}(\omega') \right] = 0 ,
\]

(12)

\[
\left[ \bar{a}_{\text{in}}(\omega), \bar{a}_{\text{in}}^\dagger(\omega') \right] = \delta(\omega - \omega') .
\]

This is really an approximate form, valid only for the case that one is dealing with a very narrow band of frequencies around a high frequency, which is always the case in quantum optics. In a future paper, we will show this is not an essential simplification. Rearranging to eliminate the internal mode,

\[
\bar{a}_{\text{out}}(\omega) = [A + (\frac{1}{2} \gamma + i\omega) \mathbb{1}]^{-1} \bar{a}_{\text{in}}(\omega) .
\]

(13)

III. A ONE-SIDED CAVITY

If the systematic part of the Hamiltonian is taken as that of a free Harmonic oscillator, we get a model of a single mode in a one-sided cavity, i.e., a cavity with significant loss through only one mirror. Thus we take

\[
H_{\text{sys}} = \hbar \omega a a^\dagger .
\]

(14)

The equation for the internal-mode operator is

\[
\frac{da}{dt} = -i\omega a - \frac{\gamma}{2} a + k\gamma a_{\text{in}}
\]

(15)

which has the solution for the frequency components

\[
\bar{a}(\omega) = \frac{\gamma}{\frac{1}{2} \gamma - i(\omega - \omega_0)} k\bar{a}_{\text{in}}(\omega)
\]

(16)

which is an ideal Lorentzian with width \( \gamma / 2 \). The factor \( k \) can be seen to be just a normalization constant for the internal mode. It can be determined by the requirement that the internal mode have the usual equal-time discrete boson commutator

\[
[a(t), a^\dagger(t)] = 1.
\]

(17)

Using the commutation relations (12) we find

\[
[a(t), a^\dagger(t')] = k^2 \gamma e^{-\gamma |t - t'|} e^{-i\omega(t - t')}
\]

(18)

giving

\[
k^2 = \gamma^{-1}
\]

(19)

so that

\[
\bar{a}(\omega) = \frac{\sqrt{\gamma}}{\frac{1}{2} \gamma - i(\omega - \omega_0)} \bar{a}_{\text{in}}(\omega) .
\]

(20)

This relationship between \( k \) and \( \gamma \) is nothing other than a quantum fluctuation-dissipation theorem (on such theorems in general see, e.g., Gardiner\(^6\)). It will be shown in our forthcoming paper that (19) is quite general. It is, in any case, clear that since \( k \) is defined in terms of the boundary condition at the mirror, it should be independent of the nature of the internal system and also of any other boundaries (mirrors). For the outgoing field one may then use the frequency-space equivalent of (5) to give

\[
\bar{a}_{\text{out}}(\omega) = \sqrt{\gamma} \bar{a}(\omega) - \bar{a}_{\text{in}}(\omega)
\]

(21)

\[
= \frac{1}{2} \gamma + i(\omega - \omega_0)
\]

(22)

\[
\bar{a}_{\text{out}}(\omega) = \frac{1}{2} \gamma - i(\omega - \omega_0)
\]

that is, the output field differs from the input by a frequency-dependent phase shift and the "out" commutation relations are the same as those for the input. Though simpler in form, this is essentially the same result as obtained by Yurke and Denker\(^7\) for signals along a transmission line connected to an \( LC \) circuit.

IV. A TWO-SIDED CAVITY

In most cavities there is a possibility of input and output in two directions. Using \( H_{\text{sys}} \) as in (14) and generalizing (6) to allow for a second external field gives

\[
\frac{da}{dt} = -i \frac{\hbar}{\gamma} [a, H_{\text{sys}}] - \frac{\gamma_1}{2} a + \frac{\gamma_2}{2} a + \sqrt{\gamma_1} a_{\text{in}} + \sqrt{\gamma_2} b_{\text{in}} .
\]

(23)

In frequency space we obtain

\[
\bar{a}(\omega) = \frac{\sqrt{\gamma_1} \bar{a}_{\text{in}}(\omega) + \sqrt{\gamma_2} b_{\text{in}}(\omega)}{\frac{1}{2} \gamma_1 + \frac{1}{2} \gamma_2 - i(\omega - \omega_0)} .
\]

(24)

The output field components \( \bar{a}_{\text{out}}(\omega) \) are then

\[
\bar{a}_{\text{out}}(\omega) = \sqrt{\gamma_1} \bar{a}(\omega) - \bar{a}_{\text{in}}(\omega)
\]

\[
= \frac{1}{2} \gamma_1 - \frac{1}{2} \gamma_2 + i(\omega - \omega_0) \bar{a}_{\text{in}}(\omega) + \sqrt{\gamma_1} \bar{b}_{\text{in}}(\omega)
\]

\[
= \frac{1}{\gamma_1 + \frac{1}{2} \gamma_2 - i(\omega - \omega_0)} .
\]

(25)
If $\gamma_2$ becomes very small we regain the results for the single-ended cavity, that is to say, a small loss has a comparably small effect on the system. If, on the other hand, the two mirrors are the same, $\gamma_1 = \gamma_2 = \gamma$, the result is

$$\tilde{a}_{\text{out}}(\omega) = \frac{i(\omega - \omega_0)\tilde{a}_{\text{in}}(\omega) + \gamma\tilde{b}_{\text{in}}(\omega)}{\gamma - i(\omega - \omega_0)}.$$  \hspace{1cm} (26)

Near resonance ($\omega \approx \omega_0$) this is approximately a through-pass Lorentzian filter

$$\tilde{a}_{\text{out}}(\omega) \approx \frac{\gamma}{\gamma - i(\omega - \omega_0)}\tilde{a}_{\text{in}}(\omega).$$  \hspace{1cm} (27)

Further away from resonance there is an increasing element of back-reflection in the output. Eventually ($|\omega - \omega_0| \gg \gamma$) the field is completely reflected,

$$\tilde{a}_{\text{out}}(\omega) \approx -\tilde{a}_{\text{in}}(\omega).$$  \hspace{1cm} (28)

A truly ideal Lorentzian through-pass filter is not, of course, possible as it would also "filter out" the commutation relations and hence the quantum noise. The element of reflection that appears in (26) is exactly sufficient for their preservation.

V. THE DEGENERATE PARAMETRIC AMPLIFIER

The systematic Hamiltonian for degenerate parametric amplification with a classical pump can be written as\(^{1,2,5}\)

$$H_{\text{sys}} = \hbar \omega_p a a + \frac{1}{2} i \hbar [e e^{-i\omega t} a + e e^{-i\omega t} a^2],$$  \hspace{1cm} (29)

where $\omega_p$ is the frequency of the pump beam and $\epsilon$ a measure of the effective pump intensity. For now the pump and cavity will be considered to be tuned so that $\omega_p = 2\omega_0$; analysis of the effect of finite detuning is postponed for a later work.

$$\tilde{a}(\omega_0 + \omega) = \frac{1}{2} \sqrt{\gamma_1 + \frac{1}{2} \gamma_2 - i \omega} \left[ \sqrt{\gamma_1} \tilde{a}_{\text{in}}(\omega_0 + \omega) + \sqrt{\gamma_2} \tilde{b}_{\text{in}}(\omega_0 + \omega) \right] + \epsilon \sqrt{\gamma_1} \tilde{a}_{\text{in}}^\dag(\omega_0 - \omega) + \sqrt{\gamma_2} \tilde{b}_{\text{in}}^\dag(\omega_0 - \omega).$$  \hspace{1cm} (36)

If both the input fields are vacuum or coherent, they will have zero normally ordered variance, that is,

$$\mathcal{C}_N(a_{\text{in}}, a_{\text{in}}^\dag) = \mathcal{C}_N(b_{\text{in}}, b_{\text{in}}^\dag) = 0,$$  \hspace{1cm} (37)

where

$$\mathcal{C}_N(a, a^\dag) = \begin{bmatrix} \langle a, a \rangle & \langle a, a^\dag \rangle \\ \langle a^\dag, a \rangle & \langle a^\dag, a^\dag \rangle \end{bmatrix}$$  \hspace{1cm} (38)

using the notation

$$\langle a, b \rangle = \langle ab \rangle - \langle a \rangle \langle b \rangle.$$  \hspace{1cm} (39)

In this case the only contribution to the normally ordered variance of the internal field will be from the commutator terms, giving

$$\langle \tilde{a}(\omega_0 + \omega) \tilde{a}(\omega_0 + \omega') \rangle = \frac{1}{2} \sqrt{\gamma_1 + \frac{1}{2} \gamma_2 - i \omega} \left[ \frac{1}{2} \sqrt{\gamma_1 + \frac{1}{2} \gamma_2 - |\epsilon|^2 + \omega^2} - \frac{1}{2} \sqrt{\gamma_1 + \frac{1}{2} \gamma_2 + |\epsilon|^2 + \omega^2} \right] \delta(\omega + \omega'),$$  \hspace{1cm} (40)

$$\langle \tilde{a}^\dag(\omega_0 + \omega) \tilde{a}(\omega_0 + \omega') \rangle = \frac{1}{2} |\epsilon|^2 \left[ \frac{1}{2} \sqrt{\gamma_1 + \frac{1}{2} \gamma_2 - |\epsilon|^2 + \omega^2} - \frac{1}{2} \sqrt{\gamma_1 + \frac{1}{2} \gamma_2 + |\epsilon|^2 + \omega^2} \right] \delta(\omega - \omega').$$  \hspace{1cm} (41)
For the full internal mode, the variances are then

$$\langle a^+, a \rangle = \frac{1}{2\pi} \int d\omega d\omega' \langle \bar{a}^+(\omega + \omega), \bar{a}(\omega + \omega') \rangle$$

$$= \frac{1}{2} \left( \frac{\gamma_1}{\gamma_1 + \gamma_2} \right)^2 \left( \frac{\gamma_2}{\gamma_1 + \gamma_2} \right)^2 |\epsilon| \langle \bar{a}^+(\omega + \omega), \bar{a}(\omega + \omega') \rangle$$

$$= \frac{1}{2} \left( \frac{\gamma_1}{\gamma_1 + \gamma_2} \right)^2 \left( \frac{\gamma_2}{\gamma_1 + \gamma_2} \right)^2 |\epsilon| \langle \bar{a}^+(\omega + \omega), \bar{a}(\omega + \omega') \rangle$$

(41)

$$\langle a^+, a \rangle = \frac{1}{2\pi} \int d\omega d\omega' \langle \bar{a}(\omega + \omega), \bar{a}(\omega + \omega') \rangle$$

$$= \frac{1}{2} \left( \frac{\gamma_1}{\gamma_1 + \gamma_2} \right)^2 \left( \frac{\gamma_2}{\gamma_1 + \gamma_2} \right)^2 |\epsilon| \langle \bar{a}(\omega + \omega), \bar{a}(\omega + \omega') \rangle$$

(42)

To see the squeezing, the field must be expressed in terms of the quadrature phases,\(^1\)\(^2\)\(^5\) Hermitian operators defined by

$$a = e^{i\theta/2}(X_1 + iX_2).$$

(43)

The normally ordered variances of these operators are

$$\langle X_1, X_1 \rangle = \frac{1}{4} \gamma_1 \gamma_2 \langle |\epsilon| \rangle,$$

$$\langle X_2, X_2 \rangle = \frac{1}{4} \gamma_1 \gamma_2 \langle |\epsilon| \rangle,$$

$$\langle X_1, X_2 \rangle = 0.$$

(44)

Perfect squeezing in one quadrature, corresponding to an eigenstate of the quadrature phase operator, is achieved with a normally ordered variance of \(-\frac{\gamma_1}{4}\). The best that can be achieved in the case of the parametric amplifier in a cavity is when on oscillation threshold, giving

$$\langle X_2, X_2 \rangle = -\frac{\gamma_1}{8}.$$  

(45)

Squeezing by a factor of one-half can thus be obtained in the \(X_2\) quadrature of the generating cavity with the \(X_1\) quadrature infinitely unsqueezed. Note that the properties of the internal mode considered in this section depend only on the total damping, not on the damping through each mirror separately. This is naturally only true when the two input fields have, as assumed, identical statistics.

The degenerate parametric amplifier in a cavity has been analyzed in depth by Milburn and Walls\(^5\) including a quantized pump beam. It is not difficult to check that their result is identical with ours for the intracavity properties, but of course it does not give any answer for the output fields.

### VI. THE OUTPUT FIELD

The internal-mode operators having already been found in terms of those for the input field, the output operators can now be calculated with use of Eq. (21):

$$\langle a^+, a_{\text{out}} \rangle$$

$$= \frac{1}{2\pi} \int d\omega d\omega' \langle \bar{a}^+(\omega + \omega), \bar{a}_{\text{out}}(\omega + \omega') \rangle$$

$$= \gamma_1 \langle a^+, a \rangle,$$

(48)

$$\langle a_{\text{out}}, a_{\text{out}} \rangle$$

$$= \frac{1}{2\pi} \int d\omega d\omega' \langle \bar{a}_{\text{out}}(\omega + \omega), \bar{a}_{\text{out}}(\omega + \omega') \rangle$$

$$= \gamma_1 \langle a^+, a \rangle.$$

(49)

Once again it is the variances in the quadrature phases which are of most interest. Defining the output quadrature phases in the same fashion as the internal ones, from (47)

$$\langle \bar{X}_{1, \text{out}}(\omega + \omega), \bar{X}_{1, \text{out}}(\omega + \omega') \rangle$$

$$= \frac{|\epsilon|}{\gamma_1} \frac{\gamma_1}{\gamma_1 + \gamma_2} \langle |\epsilon| \rangle \delta(\omega + \omega'),$$

(49)

$$\langle \bar{X}_{2, \text{out}}(\omega + \omega), \bar{X}_{2, \text{out}}(\omega + \omega') \rangle$$

$$= -\frac{|\epsilon|}{\gamma_1} \frac{\gamma_1}{\gamma_1 + \gamma_2} \langle |\epsilon| \rangle \delta(\omega + \omega'),$$

(49)
while from (48)
\[
\langle X_{1,\text{out}}, X_{1,\text{out}} \rangle = \frac{\gamma_1}{4} \frac{|\epsilon|}{\gamma_1 + \frac{1}{2} \gamma_2 - |\epsilon|},
\]
\[
\langle X_{2,\text{out}}, X_{2,\text{out}} \rangle = -\frac{\gamma_1}{4} \frac{|\epsilon|}{\gamma_1 + \frac{1}{2} \gamma_2 + |\epsilon|}.
\]

The maximum squeezing is still attained at threshold, \( |\epsilon| = \frac{1}{2} (\gamma_1 + \gamma_2) \), giving
\[
\langle \vec{X}_{2,\text{out}}(\omega_2 + \omega'), \vec{X}_{2,\text{out}}(\omega_2 + \omega') \rangle = -\frac{\gamma_1}{4} \frac{\gamma_1 + \gamma_2}{(\gamma_1 + \gamma_2)^2 + \omega^2} \delta(\omega - \omega'),
\]
\[
\langle X_{2,\text{out}}, X_{2,\text{out}} \rangle = -\frac{\gamma_1}{8}.
\]

The most interesting thing about these results is that while the squeezing in the total field is independent of \( \gamma_2 \), this is not the case for the individual frequency components. The \( \delta \) function in (51) can be removed by integrating over \( \omega' \) to give the normally ordered spectrum of the operator \( X_{2,\text{out}} \):
\[
\mathcal{S}_{2,\text{out}}(\omega_2 + \omega) = -\frac{\gamma_1}{4} \frac{\gamma_1 + \gamma_2}{(\gamma_1 + \gamma_2)^2 + \omega^2},
\]
which is a convenient way of describing the squeezing in the output field. It may be thought of loosely as the squeezing at a particular frequency, although in this case it results from the coupling of pairs of frequencies on either side of resonance: This spectrum is, ignoring the sign, a Lorentzian with peak height \( \frac{1}{2} \left[ \gamma_1 / (\gamma_1 + \gamma_2) \right] \) and width \( \gamma_1 + \gamma_2 \). Thus for a symmetric double-ended cavity with \( \gamma = \gamma_1 = \gamma_2 \)
\[
\mathcal{S}_{2,\text{out}}(0) = -\frac{1}{4}.
\]

The resonant mode of the output field is squeezed by a factor of one-half, the same as the internal field. If, however, the cavity is single-ended, with \( \gamma_2 = 0 \),
\[
\mathcal{S}_{2,\text{out}}(0) = -\frac{1}{2}
\]
and this corresponds to Yurke's single-mode analysis. Thus our multimode analysis agrees with Yurke's result, which is correct for the case that we measure only the output field corresponding to \( \omega = 0 \). Although the preferred method of measuring squeezing is by homodyne detection, we would like here to consider the effect of passive filtering of the output, to see what the effect on the squeezing is of trying to isolate the squeezed mode by means of such a passive filter.

### VII. FILTERING OF THE OUTPUT

As a model of a passive filter, we consider passing the output through a second cavity resonant with the first, so that the system is now such that a rotator or equivalent is used to isolate the input \( a_{\text{in}} \) from any feedback effects. The output field through the filter is given by (25) as
\[
\vec{e}_{\text{out}}(\omega_2 + \omega) = \frac{(\frac{1}{2} \kappa_1 - \frac{1}{2} \kappa_2 + i \omega) \vec{e}_{\text{in}}(\omega_2 + \omega) + \sqrt{\kappa_1 \kappa_2} \vec{a}_{\text{in}}(\omega_2 + \omega)}{\frac{1}{2} \kappa_1 + \frac{1}{2} \kappa_2 - i \omega},
\]
where \( \kappa_1 \) and \( \kappa_2 \) describe the mirrors of the filtering cavity. Using \( \vec{a}_{\text{in}} = a_{\text{out}} \) and assuming \( \vec{e}_{\text{in}} \) to be a vacuum field, the variances of the output field \( \vec{c}_{\text{out}} \) can be computed, and we find for the quadrature phases of the total field
\[
\langle \vec{Y}_{1,\text{out}}, \vec{Y}_{1,\text{out}} \rangle = \frac{1}{4} \frac{\kappa_1 \kappa_2 |\epsilon| \gamma_1}{(\frac{1}{2} \kappa_1 + \frac{1}{2} \kappa_2)(\frac{1}{2} \gamma_1 + \frac{1}{2} \gamma_2 - |\epsilon|)(\frac{1}{2} \kappa_1 + \frac{1}{2} \kappa_2 + \frac{1}{2} \gamma_1 + \frac{1}{2} \gamma_2 - |\epsilon|)},
\]
\[
\langle \vec{Y}_{2,\text{out}}, \vec{Y}_{2,\text{out}} \rangle = -\frac{1}{4} \frac{\kappa_1 \kappa_2 |\epsilon| \gamma_1}{(\frac{1}{2} \kappa_1 + \frac{1}{2} \kappa_2)(\frac{1}{2} \gamma_1 + \frac{1}{2} \gamma_2 + |\epsilon|)(\frac{1}{2} \kappa_1 + \frac{1}{2} \kappa_2 + \frac{1}{2} \gamma_1 + \frac{1}{2} \gamma_2 + |\epsilon|)}.
\]

At threshold, with the generating cavity single-ended (\( \gamma_2 = 0 \)),
\[
\langle Y_{2,\text{out}}, Y_{2,\text{out}} \rangle = -\frac{1}{8} \frac{\kappa_1 \kappa_2 \gamma_1}{[\gamma_1 + \frac{1}{2} \kappa_1 + \frac{1}{2} \kappa_2]^2}.
\]

For any \( \kappa_1, \kappa_2 \) this always gives less squeezing than in the initial output field \( a_{\text{out}} \). The best that can be achieved is with \( \kappa_1 = \kappa_2 >> \gamma_1 \), giving
\[
\langle Y_{2,\text{out}}, Y_{2,\text{out}} \rangle \approx -\frac{\gamma_1}{8}.
\]

This is the same as if the filter were not there at all, which is reasonable as, for large \( \kappa_1 \) and \( \kappa_2 \), the second cavity interacts strongly with the external field over a correspondingly wide bandwidth. In the alternative limit of \( \kappa = \kappa_1 = \kappa_2 << \gamma_1 \) the squeezing reduces to
\[
\langle Y_{2,\text{out}}, Y_{2,\text{out}} \rangle \approx -\frac{\kappa}{8}.
\]

Thus a through-pass filter cannot improve the squeezing in the output field. If, however, one considers the
internal mode of the second cavity, the picture is rather different. From (24)
\[
\tilde{c}(\omega + \omega) = \frac{\sqrt{\kappa_1}\tilde{c}_{\text{in}}(\omega_1 + \omega) + \sqrt{\kappa_2}\tilde{d}_{\text{in}}(\omega_2 + \omega)}{\frac{1}{2}\kappa_1 + \frac{1}{2}\kappa_2 - i\omega},
\]
(60)
or going directly to the quadrature phases,
\[
\langle \gamma_1 \rangle = \frac{\kappa_2 |\epsilon| \gamma_1}{4 (\frac{1}{2}\kappa_1 + \frac{1}{2}\kappa_2)(\frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2 - |\epsilon|)} ,
\]
\[
\langle \gamma_2 \rangle = -\frac{1}{4} (\frac{1}{2}\kappa_2 + \frac{1}{2}\kappa_2 + \frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2 + |\epsilon|) ,
\]
(62)
On threshold with the generating cavity single-ended
\[
\langle \gamma_2 \rangle = -\frac{1}{4} \frac{\kappa_2}{\kappa_1 + \kappa_2} \frac{\gamma_1}{\gamma_1 + \frac{1}{2}(\kappa_1 + \kappa_2)} .
\]
(63)
If the second cavity is also single-ended ($\kappa_1 = 0$), and is much narrower than the first ($\kappa_2 << \gamma_1$), this becomes
\[
\langle \gamma_2 \rangle = -\frac{1}{4} ,
\]
(64)
which corresponds to perfect squeezing in the second cavity. We thus see that it is indeed possible to produce arbitrarily large squeezing inside a cavity, provided this cavity is single-ended.

VIII. CONCLUSION

In this paper we have outlined general methods of relating input, output, and internal dynamics. These methods will be developed and put on a firm theoretical foundation in a forthcoming paper. The main results are a clarification of how to calculate squeezing in a multimode situation, and the demonstration that maximal squeezing inside a cavity can be achieved.

The results presented here depend on the model of a cavity mode as a harmonic oscillator. However, calculations by Gardiner and Savage\(^7\) have shown that exactly the same results arise by a detailed treatment of the motion of light waves through a cavity composed of genuine mirrors, and our forthcoming paper will unify these two treatments.

---

Nonclassical photon streams using rephased amplified spontaneous emission

Patrick M. Ledingham, William R. Naylor, and Jevon J. Longdell

Jack Dodd Centre for Photonics and Ultra-Cold Atoms, Department of Physics, University of Otago, Dunedin, New Zealand

Sarah E. Beavan and Matthew J. Sellars

Laser Physics Centre, RSPhysSE, Australian National University, Canberra, ACT 0200, Australia

(Received 15 February 2009; revised manuscript received 21 September 2009; published 4 January 2010)

We present a fully quantum mechanical treatment of optically rephased photon echoes. These echoes exhibit nonclassical noise due to amplified spontaneous emission; however, this noise can be seen as a consequence of the entanglement between the atoms and the output light. With a rephasing pulse one can get an “echo” of the amplified spontaneous emission, leading to light with nonclassical correlations at points separated in time, which is of interest in the context of building wide bandwidth quantum repeaters. We also suggest a wideband version of DLCZ protocol based on the same ideas.

DOI: 10.1103/PhysRevA.81.012301 PACS number(s): 42.50.Ex, 03.67.--a, 32.80.Qk

I. INTRODUCTION

In order to extend the range of quantum key distribution, quantum networks, and tests of Bell inequalities, a method for efficiently generating entanglement over large distances is required. To achieve this goal a quantum repeater is necessary [1]. Such repeaters are generally based on methods for entangling one light field entangled with another at a later point in time. This has led to increasing interest in quantum memories for light, which in conjunction with pair sources would achieve this. Many impressive experiments have been performed in the area of quantum memories and repeaters. The quantum state of a light field has been stored in a vapour cell with high fidelity and then measured at a later time [2]. Single photons and squeezed states have been stored and recalled [3–5], and nonclassical interference of the light from distant ensembles has been observed [6]. Entanglement [7] of and teleportation [8] between two distant trapped ions has been achieved using an optical channel, using Duan-Lukin-Cirac-Zoller- (DLCZ) type measurement-induced entanglement.

Photon echoes have a long history of use in classical signal processing with light [9,10]. There are now a number of proposals and experiments [11–16] related to the development of photon echo based quantum memories. A distinct advantage of echo based techniques is that they are multimode [17].

Current photon echo quantum memory techniques, all involve some modification of the inhomogeneous broadening profile. This imposes limits on the range of suitable materials. It would be much more convenient to use something akin to the standard two pulse echo as a quantum memory which does not require such modification.

The article is arranged in three sections. In Sec. II we present the quantum mechanical Maxwell-Bloch equations for atomic and photonic fields. Then in Sec. III, as an example of this formalism, we present an analysis of the standard two-pulse photon echo and its applicability as a quantum memory. The two-pulse echo as a quantum memory has already been investigated by others [18]. We revisit the problem here and show that this protocol fails as a quantum memory due to the strong rephasing pulse, which inverts the medium and causes additional noise on the output photonic fields. Finally in Sec. IV, after exploring the origin of this noise, we propose that this noise can be rephased to lead to time separated, temporally multimode, wide bandwidth photon streams with nonclassical correlations. So while a standard two pulse echo fails as a quantum memory, rephased amplified spontaneous emission (RASE) can be utilized in the DLCZ protocol [19]. With this modified DLCZ protocol, the inhomogeneous broadening no longer limits the time between the write and read pulses but instead increases the bandwidth of the process. This is of significance to current experiments, where the inhomogeneous broadening is an issue [20–24].

II. QUANTIZED MAXWELL-BLOCH EQUATIONS

We shall model an inhomogeneously broadened collection of two level atoms interacting with a 1D field propagating in one direction, with the following quantum Maxwell-Bloch equations:

\[
\frac{\partial}{\partial t} \hat{\sigma}_-(z, \Delta, t) = i \Delta \hat{\sigma}_-(z, \Delta, t) - i \hat{a}(z, t) \hat{\sigma}_+(z, \Delta, t)
\]

(1)

\[
\frac{\partial}{\partial t} \hat{\sigma}_+(z, \Delta, t) = i \hat{a}(z, t) \hat{\sigma}_-(z, \Delta, t) - i \hat{a}^\dagger(z, t) \hat{\sigma}_+(z, \Delta, t)
\]

(2)

\[
\frac{\partial}{\partial z} \hat{a}(z, t) = \frac{i \alpha}{2\pi} \int_{-\infty}^{\infty} \hat{\sigma}_-(z, \Delta, t) d\Delta,
\]

(3)

where \(\hat{\sigma}_{\pm,\mp}\) represent the quantum atomic spin operators, \(\hat{a}\) is the quantum optical field operator, and \(\alpha\) is the optical depth parameter, which depends on the coupling between the atoms and the field and on the atom density. The parameter \(\Delta\) is the detuning from some chosen resonant frequency and \(z\) is the distance along the propagation direction. The operators have the following commutation relations:

\[
[\hat{a}(z, t), \hat{a}^\dagger(z', t')] = \delta(t - t')
\]

(4)

\[
[\hat{\sigma}_1(z, \Delta, t), \hat{\sigma}_j(z', \Delta', t)] = \frac{2\pi}{\alpha} \epsilon_{ijk} \hat{\sigma}_k(z, \Delta, t) \delta(z - z') \delta(\Delta - \Delta').
\]

(5)
As can be seen from Eq. (3), we take the density of atoms as a function of frequency to be a constant. In the case of rare-earth ion dopants, the inhomogeneous broadening can be many times larger than the homogeneous linewidths, and as a result in most experiments without holeburnt features this is a good approximation.

The above Maxwell-Bloch equations can be derived by dividing the atomic ensemble into thin slices and then modeling each slice as a small collection of atoms inside a Fabry-Perot cavity, using standard input-output theory [25]. Taking the limit as reflectivity of the mirrors go to zero one arrives at Eqs. (1–3), where $\hat{\sigma}_z$ is the input field at the left-hand side of the cavity and $\hat{a}(z, t)$ is the output field at the right-hand side of the cavity.

### III. THE TWO-PULSE PHOTON ECHO

The first application of our quantum Maxwell-Bloch equations will be in analyzing a memory based on a two pulse photon echo. The Maxwell-Bloch equations are nonlinear and in general difficult to solve analytically; however, following work done with the semiclassical Maxwell-Bloch equations [26] one can make reasonable approximations that simplify the situation greatly. These approximations are illustrated in Fig. 1. First we shall assume the input pulse is weak and is much smaller than a π pulse. In this case all the atoms will stay near their ground state ($\sigma_+ \approx +1$) and we can approximate the atomic lowering operator $\sigma_-$ as a harmonic oscillator field $D_\sigma$. The result are linear equations that we shall refer to as the ground-state Maxwell-Bloch equations,

$$\frac{\partial}{\partial t} \hat{D}_\sigma(z, \Delta, t) = i \Delta \hat{D}_\sigma(z, \Delta, t) + i \hat{a}(z, t), \quad (6)$$

$$\frac{\partial}{\partial z} \hat{a}(z, t) = \frac{i \alpha}{2\pi} \int_{-\infty}^{\infty} \hat{D}_\sigma(z, \Delta, t) d\Delta. \quad (7)$$

Equation (6) is just a first-order linear equation with solution,

$$\hat{D}_\sigma(z, \Delta, t) = -i \int_{-\infty}^{t} dt' \hat{a}(z, t') e^{i\Delta(t-t')} + e^{i\Delta t} \hat{D}_\sigma^0(z, \Delta), \quad (8)$$

where $\hat{D}_\sigma^0(z, \Delta)$ is an initial condition. Taking the Fourier transform of Eqs. (7) and (8) and substituting, one arrives at the following expression:

$$\frac{\partial}{\partial z} \hat{a}(z, \omega) = -\frac{\alpha}{2\pi} \int_{-\infty}^{\infty} d\Delta \hat{a}(z, \omega) \left[ \frac{1}{i(\omega - \Delta)} + \pi \delta(\omega - \Delta) \right]$$

$$+ \frac{i \alpha}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\Delta \delta(\omega - \Delta) \hat{D}_\sigma^0(z, \Delta),$$

$$= -\frac{\alpha}{2} \hat{a}(z, \omega) + \frac{i \alpha}{\sqrt{2\pi}} \hat{D}_\sigma^0(z, \omega), \quad (9)$$

where $\delta(\omega)$ is the Dirac delta function. Solving Eq. (9) and Fourier transforming back to the time domain we get

$$\hat{a}(z, t) = \hat{a}(0, t) e^{\alpha z^2/2} + \frac{i \alpha}{\sqrt{2\pi}} \int_{0}^{z} \hat{a}' e^{i\alpha(z-z')^2} \hat{D}_\sigma^0(z', t), \quad (10)$$

where $\hat{a}(0, t)$ denotes the input photonic field. Equations (8) and (10) form the ground-state solutions for all input times.

After the π pulse the atoms are all very close to the excited state ($\sigma_+ \approx +1$) in which case we can approximate $\sigma_+$ by a harmonic oscillator field $D_\sigma$. This gives us the excited-state Maxwell-Bloch equations.

$$\frac{\partial}{\partial t} \hat{D}_\sigma^1(z, \Delta, t) = i \Delta \hat{D}_\sigma^1(z, \Delta, t) - i \hat{a}(z, t), \quad (11)$$

$$\frac{\partial}{\partial z} \hat{a}(z, t) = \frac{i \alpha}{2\pi} \int_{-\infty}^{\infty} \hat{D}_\sigma^1(z, \Delta, t) d\Delta. \quad (12)$$

We treat the π pulse as being a perfect π leading to the transformation $D_e \leftarrow \hat{D}_\sigma$. We will discuss the treatment of the perfect π pulse later in the text.

Bringing Eqs. (11) and (12) through the same mathematical process as Eqs. (6) and (7), we arrive at the excited-state solutions:

$$\hat{D}_\sigma^1(z, \Delta, t) = i \int_{-\infty}^{t} dt' \hat{a}(z, t') e^{i\Delta(t-t')} + e^{i\Delta t} \hat{D}_\sigma^0(z, \Delta), \quad (13)$$

$$\hat{a}(z, t) = \hat{a}(0, t) e^{\alpha z^2/2} + \frac{i \alpha}{\sqrt{2\pi}} \int_{0}^{z} \hat{a}' e^{i\alpha(z-z')^2} \hat{D}_\sigma^0(z', t), \quad (14)$$

where $\hat{a}(0, t)$ and $\hat{D}_\sigma^0(z, \Delta)$ are initial conditions for the photonic and atomic excited fields, respectively.

Matching the ground [Eqs. (8) and (10)] and excited-state solutions [Eqs. (13) and (14)] at the point the π pulse is applied we get a complete solution. The efficiency is $\sinh^2(\sqrt{\alpha} z)$ and in the limit of large optical depths high efficiencies are possible. Physically, this is because the photon echo is produced in the first piece of the sample and then gets amplified as it propagates through the inverted medium. The noise on the output can be quantified by considering the case of no input pulse, and then the output will be amplified spontaneous emission (ASE), simply the vacuum noise amplified by the gain of $\exp(\alpha z)$ of the inverted ensemble. In the case of no input pulse, we get an incoherent output field with $\langle \hat{a}'(t') \hat{a}(t') \rangle = \delta(t-t') \exp(\alpha l) - 1$. It is interesting to consider the source of this noise. In the model we have no dissipation and so the total system evolves through pure states.

Equation (8) is analogous the output of a beam splitter. The input fields being light and atoms, with the output fields...
consisting of combinations of photonic and atomic excitations. One can see that the addition of atomic excitations in the solution is necessary for the conservation of the commutation relations due to the input photonic field decaying away at large $\alpha l$. The excited-state solution is analogous to a nondegenerate parametric amplifier [27], here the input field is amplified, and the commutation relations are preserved by the addition of atomic creation operators. The state of one output mode is mixed only if the other is traced over; if the system is viewed as a whole one has an entangled state. In the next section we show that by applying a rephasing pulse to the ensemble we can turn the excitation of the atoms back into light, leading to streams of photons with highly nonclassical correlations between two points separated in time.

IV. REPHASED AMPLIFIED SPONTANEOUS EMISSION

Now we consider the two $\pi$ pulse sequence shown in Fig. 2. For region 1 the atoms will be inverted due to the first $\pi$ pulse and hence Eqs. (11) and (12) will apply. For region 2 the atoms will be near the ground state due to the refocusing $\pi$ pulse, hence Eqs. (6) and (7) describe the dynamics. We take the second $\pi$ pulse to occur at $t = 0$.

The solution for the light in region 1 is given by Eqs. (8) and (10) and the solution in region 2 is given by Eqs. (13) and (14). For boundary conditions we take the incident field, $\hat{a}(0,t)$, to be in its vacuum state as we do for the initial condition $D_{\text{eo}}(z,\Delta)$. The initial condition for region 2 we get from the final condition for region 1:

$$\hat{D}_{\text{eo}}(z,\Delta) = i e^{\alpha z/2} \int_{-\infty}^{0} dt' \hat{a}^\dagger(0,t') e^{i \Delta t'} + \frac{\alpha}{\sqrt{2\pi}} \int_{-\infty}^{0} dt' e^{i \Delta t'} \times \int dz' e^{i\alpha z'/2} \hat{D}_{\text{eo}}(z',t') + \hat{D}_{\text{eo}}(z,\Delta).$$

These boundary and initial conditions substituted in Eqs. (8)–(14) give a complete analytic solution of the linearised Maxwell-Bloch equations.

To show that the photon streams described by these solutions have nonclassical correlations we consider,

$$R = \frac{p(t_1, t_2)^2}{p(t_1, t_1) p(t_2, t_2)},$$

where $p(t_i, t_j) = \langle \hat{a}^\dagger(l, t_i) \hat{a}(l, t_j) \hat{a}(l, t_j) \hat{a}(l, t_i) \rangle$. For classical fields the Cauchy-Schwartz inequality states that $R \leq 1$ [27]. Considering times equally separated about the second $\pi$ pulse, for the expression for the output fields derived above we get

$$R(\alpha l) = \left[ \frac{1}{2} + \frac{\alpha l + \cosh(\alpha l)}{4 \sinh \left( \frac{\alpha l}{2} \right) \left[ e^{\alpha l} - 1 \right]} \right]^2.$$

Figure 3 shows that for small optical depths ($\alpha l < 1$) the output at times equally separated from the refocusing $\pi$ pulse has nonclassical correlations. It should be pointed out that the entanglement between times is not perfect because the echo efficiency is not 100%. The detection of a photon in Region 2 at a particular time means that there must have been one in Region 1 at the matching time; however, the converse is not true.

An advantage that RASE has is its potential implementation in a larger range of systems. This is in contrast with the implementation of current photon echo quantum memories. Current quantum memory echo techniques use very fine spectral features prepared in the inhomogeneous line rather than the natural inhomogeneous profile and optical rephasing pulses. Indeed the fact that AFC [15] and CRIB [11,12] type echoes ideally want homogeneously broadened ensembles has led to their investigation in non-solid-state systems [28]. When selecting a rare-earth ion system, one finds that the systems long-lived spectral holes, such as europium or praseodymium, have inconvenient wavelengths ($\approx$580 nm and $\approx$696 nm), in systems which are much more compatible with optical fibers and diode lasers the holes are much more transient, making high-fidelity operation difficult at best.

V. IMPERFECT $\pi$ PULSES

So far we have treated the $\pi$ pulses as ideal and the effect of nonideal $\pi$ pulses needs to be considered. It is feasible to make a pulse such that afterward one can make the approximation $\sigma_z \approx 1$ especially as we are interested in optically thin samples. The ability to do this in optically thick samples is also helped by the area theorem, which states that a $\pi$ pulse remains a $\pi$ pulse as it propagates through a medium [18,29]. In the situation where $\sigma_z \approx 1$ after the pulse, we can model our nonideal $\pi$ pulse as the combination of an ideal $\pi$ pulse and some excitation of the $\hat{D}_s$ field. This excitation of the $\hat{D}_s$ field will be temporally brief, and if the inhomogeneous broadening is flat the ensemble of atoms will quickly dephase leading to...
no net polarization in the ensemble shortly after the $\pi$ pulse. This means that the excitation produced by the imperfect $\pi$ pulse will no longer interact with the optical field (unless it is rephased by another strong pulse). The ability to prepare an ideal inverted medium for classical information processing has been investigated experimentally [30].

VI. PHASE MATCHING

The treatment so far has involved only one spatial dimension. One way to consider the effect of phase matching is by extending to a 3D treatment in the paraxial approximation. In this case we replace $a(z, t) \rightarrow a(z, k, t)$ and $\sigma_\pm(z, \Delta, t) \rightarrow \sigma_\pm(z, \rho, \Delta, t)$, where $\mathbf{k} = (k_x, k_y)$ is the transverse wave vector and $\rho = (x, y)$ is the transverse position. Our linearized Maxwell Bloch equations for the ground state become

$$\frac{\partial}{\partial t} \hat{D}_g(z, \rho, \Delta, t) = i \Delta \hat{D}_g(z, \rho, \Delta, t) + \frac{i}{4\pi} \int d^2k \hat{a}(z, k, t) e^{i \mathbf{k} \cdot \rho}. \quad (18)$$

$$\frac{\partial}{\partial z} \hat{a}(z, k, t) = \frac{i \alpha}{2\pi} \int d^2 \rho \int_{-\infty}^{\infty} d \Delta \hat{D}_g(z, \rho, \Delta, t) e^{-i \mathbf{k} \cdot \rho}. \quad (19)$$

Fourier transforming the atomic operators along the transverse dimensions by defining

$$\hat{D}_g(z, k_x, \Delta, t) = \int d^2 \rho \hat{D}_g(z, \rho, \Delta, t) \exp(-i \mathbf{k} \cdot \rho) \quad (20)$$

leads to Maxwell-Bloch equations that are diagonal in the transverse wave vector

$$\frac{\partial}{\partial t} \hat{D}_g(z, k_x, \Delta, t) = i \Delta \hat{D}_g(z, k_x, \Delta, t) + i \hat{a}(z, k_x, t) \quad (21)$$

$$\frac{\partial}{\partial z} \hat{a}(z, k_x, t) = \frac{i \alpha}{2\pi} \int_{-\infty}^{\infty} d \Delta \hat{D}_g(z, k_x, \Delta, t). \quad (22)$$

For the excited-state Maxwell-Bloch equation, the same procedure gives

$$\frac{\partial}{\partial t} \hat{D}_e(z, k_x, \Delta, t) = i \Delta \hat{D}_e(z, k_x, \Delta, t) - i \hat{a}(z, -k_x, t) \quad (23)$$

$$\frac{\partial}{\partial z} \hat{a}(z, k_x, t) = \frac{i \alpha}{2\pi} \int_{-\infty}^{\infty} \hat{D}_e(z, -k_x, \Delta, t) d \Delta. \quad (24)$$

In the situation where the $\pi$ pulse is applied off axis the phase of the $\pi$ pulse depends on the transverse position leading to the transformation

$$\hat{D}_e(z, \rho, \Delta, t) \leftrightarrow \hat{D}_e(z, \rho, \Delta, t) \exp(2i(k_x \cdot \rho)). \quad (25)$$

or, after Fourier transforming,

$$\hat{D}_e(z, k_x, \Delta, t) \leftrightarrow \hat{D}_e(z, k_x - 2k_x, \Delta, t). \quad (26)$$

With the RASE pulse sequence described in Fig. 2, the phase of the first $\pi$ pulse does not matter. The atoms are all in the ground state before the pulse and assuming a perfect $\pi$ pulse will end up in the excited state afterward regardless. Any small coherent excitation caused by imperfect $\pi$ pulses can be ignored; it will quickly dephase because of the inhomogeneous broadening and will not be rephased as an echo until after the second $\pi$ pulse that is outside the region of time of interest. The ASE caused by the inversion due to this $\pi$ pulse will be spatially multimode, with the amount of ASE in a particular mode determined by the gain experienced traversing the sample.

Suppose we set a detection system to look at the ASE produced with wave vector $\mathbf{k}_{\text{ASE}}$, from Eqs. (23) and (24). We can see that the light with this wave vector is entangled with the atomic excitation with mode $-\mathbf{k}_{\text{ASE}}$. The $\pi$ pulse transfers this to the wave vector $-\mathbf{k}_{\text{ASE}} + 2\mathbf{k}_\pi$ according to Eq. (26). Equations (21) and (22) connect atomic and optical modes with the same wave vector so we have that the wave vector for the RASE is $\mathbf{k}_{\text{RASE}} = -\mathbf{k}_{\text{ASE}} + 2\mathbf{k}_\pi$ or

$$\mathbf{k}_{\text{ASE}} + 2\mathbf{k}_\pi = 2\mathbf{k}_\pi. \quad (27)$$

This is the same phase-matching condition as a two-pulse photon echo, $\mathbf{k}_{\text{input}} + \mathbf{k}_{\text{echo}} = 2\mathbf{k}_\pi$. While this phase-matching condition is valid outside the paraxial regime, the only way to achieve phase matching is with the beams collinear or close to collinear, because the ASE the RASE and the $\pi$ pulse must all be at the same frequency.

A. DLCZ Protocol

It is interesting to consider the relationship of the current scheme with the DLCZ protocol [19]. The DLCZ protocol involves the creation of entanglement between distant ensembles. The relevant energy level diagrams are shown in Fig. 4(a). Once the level $\mid 3 \rangle$ has been adiabatically eliminated, the write process is formally equivalent to a set of excited-state atoms ($\mid 1 \rangle$) spontaneously emitting into the level $\mid 2 \rangle$. The emitted optical field is then steered elsewhere for entanglement generation with another ensemble of atoms [19]. Once entanglement is generated between two ensembles, one wishes to read out one ensemble atomic field to a photonic field in order to implement entanglement swapping [19]. For the read process, state $\mid 2 \rangle$ becomes the excited state and state $\mid 1 \rangle$ the ground state.

![FIG. 4.](image-url)

FIG. 4. (Color online) (a) DLCZ protocol showing write and read process. (b) Modified protocol. The inhomogeneous broadening of the $\mid 1 \rangle \rightarrow \mid 2 \rangle$ transition now leads to an increase in bandwidth.
One problem with this is the inhomogeneous broadening of the |1⟩-|2⟩ transition causes dephasing limiting the time separation between the writing and reading process. A modified DLCZ protocol, in close analogy with RASE, would overcome this problem. A rephasing pulse on the |1⟩-|2⟩ transition utilizes the inhomogeneous broadening, now increasing the bandwidth of the process rather than reducing the time separations. The sequence of events for this modified DLCZ protocol are shown in Fig. 4(b). It is worth noting that the modified DLCZ protocol does not have the same issue with echo efficiency as the two-level scheme because the classical coupling field can be altered meaning that the ensemble can be optically thin for the writing process and thicker for the reading process.

The phase-matching conditions for the modified DLCZ protocol will be the same as given in Eq. (27) for RASE.

However, with a Raman transition it is the wave vector difference for the two optical fields that is important. This means that one has alot more freedom in the implementation because one is not restricted by the requirement that $\omega = ck$, as one is in the two-level case.

VII. CONCLUSION

In conclusion we have shown that rephased amplified spontaneous emission has strongly nonclassical correlations with the original amplified spontaneous emission in the optically thin regime. This leads to the possibility of a modified DLCZ protocol, where the problem of dephasing due to inhomogeneous broadening of the hyperfine transitions is solved by a rephasing pulse, increasing the bandwidth of the process.
Efficient Quantum Memory Using a Weakly Absorbing Sample

Mahmood Sabooni, Qian Li, Stefan Kröll, and Lars Rippe
Department of Physics, Lund University, P. O. Box 118, SE-22100 Lund, Sweden
(Received 29 December 2012; published 26 March 2013)

A light-storage experiment with a total (storage and retrieval) efficiency $\eta = 56\%$ is carried out by enclosing a sample, with a single-pass absorption of $10\%$, in an impedance-matched cavity. The experiment is carried out using the atomic frequency comb (AFC) technique in a praseodymium-doped crystal ($0.05\%$Pr$^{3+}$:Y$_2$SiO$_5$) and the cavity is created by depositing reflection coatings directly onto the crystal surfaces. The AFC technique has previously by far demonstrated the highest multimode capacity of all quantum memory concepts tested experimentally. We claim that the present work shows that it is realistic to create efficient, on-demand, long storage time AFC memories.

A quantum memory that has the ability to map onto, store in, and later retrieve the quantum state of light from matter is an important building block in quantum information processing [1]. Quantum memories are expected to become vital elements for long distance quantum key distribution [2,3]. Quantum computing based on linear optics schemes [4], signal synchronization in optical quantum processing [5,6], the implementation of a deterministic single-photon source [7], and precision measurements based on mapping of the quantum properties of an optical state to an atomic ensemble [8] are other applications of quantum memories. For most of the applications mentioned, high performance will be required in terms of high efficiency [9,10], on-demand readout, long storage time [11,12], multimode storage capacity [13,14], and broad bandwidth [15].

Many protocols have been proposed to realize an efficient quantum memory; these include electromagnetically induced transparency (EIT) [16], off-resonant Raman interactions [17], controlled reversible inhomogeneous broadening (CRIB) [18–20], gradient echo memory (GEM) [21], and atomic frequency combs (AFC) [22]. The most impressive storage and retrieval efficiencies so far, $87\%$ [9] and $69\%$ [10], were achieved in hot atomic vapor and rare-earth doped crystals, respectively, using the GEM technique. Additionally, $43\%$ storage and retrieval efficiency using EIT in a hot Rb vapor [23] and $35\%$ using AFC in a rare-earth doped crystal [24] were attained.

The AFC technique [22] is employed in this Letter because the number of (temporal) modes that can be stored in a sample is independent of the optical depth ($d$) of the storage material, in contrast to other quantum memory approaches. An AFC structure consists of a set of (artificially created) narrow absorbing peaks with equidistant frequency spacing $\Delta$ and uniform peak width $\gamma$ (see the inset in Fig. 3). An input (storage) field (at time $t = 0$) that spectrally overlaps the AFC structure will be absorbed and leave the absorbers (in our case the Pr ions) in a superposition state [22]. If the coherence time is long compared to $1/\Delta$, a collective emission due to constructive interference (similar as for the modes in a mode-locked laser) will occur at time $t_{\text{echo}} = 1/\Delta$.

High storage and retrieval efficiency is one of the main targets of quantum memories and this relies on strong coupling between light and matter [1]. One approach for studying light-matter interaction is based on the high finesse cavity-enhanced interaction of light with a single atom [25,26]. Another alternative for increasing the coupling efficiency of a quantum interface between light and matter is using an optically thick free space atomic ensemble [1]. In this Letter, we combine the advantages of both approaches to implement an efficient quantum interface in a weakly absorbing solid state medium [27]. Within the ensemble approach several experimental realizations from room-temperature alkali gases [28], to alkali atoms cooled and trapped at temperature of a few tens or hundreds of microkelvin [29] have been investigated. Among the ensemble-based approaches impurity centers in a solid state crystal is a powerful alternative for quantum memories because of the absence of atomic movement.

The objective of this Letter is to demonstrate a quantum memory with high storage and retrieval efficiency, with the added benefit of being achievable in a weakly absorbing medium. Another benefit is the short crystal length ($2\,$mm), and small physical storage volume ($\ll \text{mm}^3$). This can simplify the implementation of long-term storage in the hyperfine levels, as will be further discussed at the end of the Letter. For an AFC memory with no cavity and with emission in the forward direction, the maximum theoretical efficiency is limited to $54\%$ [22]. Our work is based on the proposals in Refs. [30,31], where it is shown that close to unity storage and retrieval efficiency can be obtained, using an atomic ensemble in an impedance-matched cavity. A cavity can be impedance matched, by having the absorption per cavity round trip ($1 - e^{-2\alpha L}$) equal to the transmission of the input coupling mirror ($1 - R_1$), while the back mirror is $100\%$ reflecting, which gives $R_1 = e^{-2\alpha L}$, where $\alpha$ is the absorption coefficient and $L$ is the
length of the optical memory material. For this impedance-matching condition, all light sent to the cavity will be absorbed in the absorbing sample inside the cavity and no light is reflected.

The storage cavity is made up of a 2 mm long 0.05% Pr\(^{3+}:\text{Y}_2\text{SiO}_5\) crystal; see Fig. 1. To reduce the complexity of the alignment and reduce losses, the crystal surfaces are reflection coated directly, rather than using separate mirrors. The two cavity crystal surfaces are not exactly parallel as shown in Fig. 1 (\(\theta = 10\) arcsec). Part of the incoming field \(\vec{E}_{\text{in}}\) will be reflected \(\vec{E}_{\text{refl}}\) at the first mirror surface \(R1\), see Fig. 1. The field leaking out through \(R1\) from the cavity, \(\vec{E}_{\text{leak}}\), is coherently added to \(\vec{E}_{\text{refl}}\) such that \(\vec{E}_{\text{out}} = \vec{E}_{\text{refl}} + \vec{E}_{\text{leak}}\). At the impedance-matched condition, \(\vec{E}_{\text{refl}}\) and \(\vec{E}_{\text{leak}}\) differ in phase by \(\pi\) and have the same amplitude \(|\vec{E}_{\text{refl}}| = |\vec{E}_{\text{leak}}|\). This means that the light intensity at the reflection detector (PD3) should ideally vanish if this condition is satisfied.

The cavity crystal (impedance matching) was first tested without memory preparation. The effective cavity length should ideally vanish if this condition is satisfied. Then, a weak Gaussian pulse \((\tau_{\text{FWHM}} = 250\) ns\)) with a small pulse area and repetition rate of 10 Hz was injected into the cavity while the laser frequency was slowly scanned during \(\sim 6\) s across the inhomogeneous Pr ion absorption line. To find the best impedance-matched point, the cavity crystal was translated in small steps perpendicular to the beam direction. For each step a 18 GHz laser scan was carried out. The calibrated reflected part of the input pulse (PD3/PD1) is plotted versus the frequency offset from the Pr\(^{3+}:\text{Y}_2\text{SiO}_5\) inhomogeneous line center in Fig. 2, for the position where the highest absorption was obtained. This measurement shows that a maximum of about 84% of the input energy could be absorbed. This occurred about 45 GHz above the inhomogeneous broadening center frequency. This will set an upper limit for the achievable storage and retrieval efficiency in the present setup. Due to the absorption tailoring during the memory preparation (to be discussed below), the impedance-matching condition will be fulfilled closer to the inhomogeneous line center in the actual storage experiment, but the measurement establishes the losses caused by spatial mode mismatching. In addition, in the present setup the dye laser is frequency stabilized against a Fabry-Pérot cavity using the Pound-Drever-Hall (PDH) locking technique [32,33]. This provides more freedom for locking the laser frequency further away from the inhomogeneous profile center compared to our earlier work [34] where stabilization based on hole burning was used. This has a
large influence on improving the impedance-matching condition.

To demonstrate a quantum memory based on the AFC protocol, first, a transparent (nonabsorbing) spectral transmission window within the Pr ion absorption profile was created using optical pumping. An accurate description of the pulse sequences required for creating such a transparency window, which henceforth is called a spectral pit, is given in Ref. [24]. Because of the strong dispersion created by the spectral pit [35], the cavity free spectral range (FSR) and the cavity linewidth are reduced by 3–4 orders of magnitude [36]. The reduction can be understood as follows. The cavity FSR is \( \Delta \nu_{\text{mode}} = \frac{c_0}{2 \pi \eta} \) [37] where \( c_0 \) is speed of light in the vacuum, \( L \) is the cavity length, and \( n_g \) is the group refractive index. \( n_g = n_r(\nu) + \nu \frac{dn_r(\nu)}{d\nu} \) and \( n_r(\nu) \) is the real refractive index as a function of frequency \( \nu \). In case of no or constant absorption, the dispersion term is negligible compared to the real refractive index \( (n_r(\nu) \gg \nu \frac{dn_r(\nu)}{d\nu}) \). The FSR of this cavity with no absorbing material is about 40 GHz. In the presence of sharp transmission structures \( (n_r(\nu) \ll \nu \frac{dn_r(\nu)}{d\nu}) \) a dramatic reduction of the cavity FSR and the cavity linewidth can occur. In our case \( \nu \frac{dn_r(\nu)}{d\nu} > 1000 n_g(\nu) \) and the reduction is \( >3 \) orders of magnitude at the center of the inhomogeneous line. A more detailed description of the cavity FSR reduction is given in Refs. [34,36]. Translating the crystal perpendicular to the beam propagation direction will move the cavity transmission within the spectral pit due to the small wedge on the crystal.

After preparing the transparent (nonabsorbing) spectral transmission window, each AFC peak is created using a complex hyperbolic secant pulse (sechyp for short) with chirp width \( \tilde{f}_{\text{width}} = 70 \) kHz [38] and temporal width \( \tau_{\text{FWHM}} = 16.8 \mu s \) [38]. This pulse excites ions from \( |\pm \frac{1}{2} g\rangle \rightarrow |\pm \frac{1}{2} e\rangle \) state [see Fig. 1(a)]. From the \( |\pm \frac{1}{2} e\rangle \) state, Pr ions will decay mostly to the \( |\pm \frac{1}{2} g\rangle \) due to the high branching ratio for the \( |\pm \frac{1}{2} e\rangle \rightarrow |\pm \frac{1}{2} g\rangle \) transition [39]. This pulse is repeated several (\( \sim 50 \)) times with a waiting time of 500 \( \mu s \) between each pulse. This process creates one AFC peak. Repeating this procedure with a consecutive change of center frequency of the sechyp pulse by \( \Delta \) will create the other AFC peaks. The finesse of the AFC structure with peak width \( \gamma \) and peak separation \( \Delta \) will be \( F_{\text{AFC}} = \frac{\Delta}{\gamma} \).

As we discussed earlier, the absorption engineering of the Pr ions inside the cavity will directly affect the cavity modes via strong dispersion. Therefore, measuring the AFC structure properties in the cavity case is challenging. In order to at least to some extent estimate the AFC structure properties, the coating of a small part of the cavity crystal is removed, and the same preparation as for the memory is performed in this part. The remaining coating could affect the precision of this measurement through the cavity dispersion effect. The trace in Fig. 3 is recorded using a weak readout beam that is frequency chirped at a rate of \( 10 \) kHz/\( \mu s \) across the frequency region of the AFC structure. The overall spectral structure is complicated and a detailed discussion of the spectrum is beyond the scope of the present Letter. The inset in Fig. 3 shows the spectral content of the storage pulse relative to the four AFC peaks. The cavity transmission linewidth is tuned to be at least as wide as the whole prepared AFC structure [34]. In this case it was \( \approx 11 \) MHz wide.

The input pulse is stored using the \( |\pm \frac{1}{2} g\rangle \rightarrow |\pm \frac{1}{2} e\rangle \) transition for ions initially in state \( |\pm \frac{1}{2} g\rangle \). A Gaussian pulse with a duration of \( \tau_{\text{FWHM}} = 250 \) ns and small pulse area is employed as a storage pulse. The frequency of the storage pulse is tuned to the center frequency of an AFC structure with a peak separation \( \Delta = 0.9 \) MHz. The retrieved echo pulse is detected at detector PD3 after 1.1 \( \mu s \) as shown with black solid line in Fig. 4. In addition, multiple echoes are detected at times 2.2 \( \mu s \) and 3.3 \( \mu s \) which are probably due to multiple rephasing of the ensemble.

In order to assess the storage and retrieval efficiency, the intensity of the input storage pulse at the cavity crystal should be measured. To this end, the cavity crystal was turned \( \sim 180^\circ \) such that the input storage pulse impinged on the \( R2 = 99.7\% \) mirror, with no pit and peak creation active. In this way the input storage pulse is (almost completely) reflected and the signal on PD3, after calibrating using PD1, can be used as a reference value for the storage pulse input intensity, as shown by a red dashed line in Fig. 4. The pulse area of the first echo at \( \approx 1.1 \) \( \mu s \)
FIG. 4 (color online). The input storage pulse as a reference detected at PD3 (see text) is shown as a red dashed line. The retrieved echo pulse is detected at detector PD3 after 1.1 µs as shown with the black solid line. The area of the echo pulse at 1.1 µs divided by the reference signal pulse area gives a storage and retrieval efficiency of the memory of $\eta = 56\%$.

divided by the reference signal pulse area gives a storage and retrieval efficiency of the memory of $\eta = 56\%$ (four independent measurements gave values ranging from 55.5% to 56.8%). The leakage through the cavity detected on PD2 (see Fig. 1) during the storage is almost negligible (≈1% of the storage pulse).

The present result is lower than the best storage and retrieval efficiency achieved elsewhere [9,10]; however, it is the highest storage and retrieval efficiency based on the AFC protocol, which is presently the best multimode quantum state storage protocol [40], also in the cavity configuration [31]. The source of losses in the current setup can be estimated as about 16% related to the impedance mismatching, 5% associated with the multiple echoes, about 1% from transmission through the cavity, about 10% linked to the AFC finesse dephasing factor, and the rest possibly related to the crystal background absorption. By addressing these losses the efficiency could be improved. We estimate that the efficiency under the same conditions as in this experiment for a crystal in a single-pass configuration would be <1% [34]. Therefore the cavity configuration shows a significant enhancement compared to the bare crystal.

In order to obtain on-demand and long-term storage based on the AFC protocol [22], the ground-excited state superposition should be transferred to, and then brought back from, a spin-level superposition between two of the ground states [41]. The dephasing due to the spin state inhomogeneous broadening can be suppressed using radio-frequency (rf) spin-echo techniques. In addition, even longer (>60s) storage time is possible by utilizing zero first-order Zeeman (ZEFOZ) shift [42] and spin-echo techniques to suppress slow variations of the spin transition frequencies [11,12]. Quantum memories in a smaller physical volumes requires significantly lower rf power. Although the present result is 13% lower than the highest rare-earth crystal efficiency results so far [10], it is obtained in a crystal that is 10 times shorter. This may in practice significantly simplify long time, high efficiency spin storage, since too large volumes will require excessive rf powers to compensate for the spin inhomogeneous broadening. In addition, efficient quantum memories in a weakly absorbing media opens up the possibility of utilizing materials with low optical depth but good coherence properties (e.g., Eu$^{3+}$:Y$_2$SiO$_5$ [43]).

In summary, we have demonstrated a quantum memory with $\eta = 56\%$ storage and retrieval efficiency based on the AFC protocol. This is done in a weakly absorbing medium and short crystal length (2 mm) by utilizing an impedance-matched cavity configuration. This achievement, in addition to the storage and recall of weak coherent optical pulses [14,44], spin-wave storage demonstration [41], the best multimode quantum memory [13,14], and storage of entanglement [15,45] increases the possibility of creating an efficient, on-demand, long storage time, and multimode quantum memory based on the AFC protocol in the future.

This work was supported by the Swedish Research Council, the Knut & Alice Wallenberg Foundation, the Maja & Erik Lindqvists forskningsstiftelse, the Crafoord Foundation, and the EC FP7 Contract No. 247743 (QuRep) and (Marie Curie Action) REA Grant No. 287252 (CIPRIS).

Cavity enhanced rephased amplified spontaneous emission

Lewis A Williamson and Jevon J Longdell
The Jack Dodd Centre for Quantum Technology, Department of Physics, University of Otago, Dunedin, New Zealand.
E-mail: jevon.longdell@otago.ac.nz

Abstract. Amplified spontaneous emission is usually treated as an incoherent noise process. Recent theoretical and experimental work using rephasing optical pulses has shown that rephased amplified spontaneous emission (RASE) is a potential source of wide bandwidth time-delayed entanglement. Due to poor echo efficiency the plain RASE protocol doesn’t in theory achieve perfect entanglement. Experiments done to date show a very small amount of entanglement at best. Here we show that rephased amplified spontaneous emission can, in principle, produce perfect multimode time-delayed two mode squeezing when the active medium is placed inside a Q-switched cavity.

PACS numbers: 03.67.-a, 32.80.Qk, 42.50 p, 78.47.jf
1. Introduction

The bitrate available from current quantum networks falls off very quickly with increasing attenuation in the transmission path. Quantum repeaters [1] stand to alleviate this problem. The proposal of Duan-Lukin-Cirac and Zoller (DLCZ) [2] first suggested the use of atomic ensembles to generate time-separated entangled photons. There has been a lot of experimental progress in the use of atomic ensembles [3, 4, 5, 6, 7, 8, 9, 10], but not yet a practical quantum repeater.

Quantum memories, like the DLCZ protocol, use the large interactions possible between an optical field and a collective excitation in an ensemble. Early work on quantum memories proposed using photon echo techniques and inhomogeneously broadened atoms to efficiently store and retrieve quantum states of light [11, 12, 13, 14] (for a review, see [15]). Since these initial proposals, there has been impressive demonstrations of quantum memories using photon echo techniques [16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. A distinct advantage such techniques have is that they are multimode [26, 27]. Ledingham et al. [28] showed how rephased amplified spontaneous emission (RASE) can be used as a source of photon streams with time-separated entanglement. This suggested that photon echoes were not only useful for making the quantum memories used in quantum repeaters; they could also be used as the entanglement source. The RASE is illustrated in figure 1.

![Figure 1](image-url)

**Figure 1.** The RASE scheme. The first $\pi$-pulse shifts the two-level atoms into their excited state. Spontaneous emission from the excited atom ensemble produces the amplified spontaneous emission (ASE). During this emission, the atoms continually dephase due to inhomogeneous broadening. The rephasing $\pi$-pulse inverts the excitation of and rephases the atoms. Atoms that contributed to the ASE continue to emit photons to produce the RASE. This results in RASE that is a time-reversed “echo” of the ASE. The RASE and ASE are correlated at times equally spaced from the rephasing $\pi$-pulse.

Realization of the RASE scheme has been demonstrated in two separate systems. One system follows the original proposal [29] and the other [30] implements a modified
approach based on four-level atoms [31]. Both demonstrations show that correlations exist between the ASE field and the RASE field. In the case of [29] the correlation was strong enough and the noise was low enough to show evidence of entanglement, although the confidence level was not high.

As a source of entanglement, RASE as it was initially proposed in [28] and [31] is imperfect. The main problem is the efficiency of the recall of entanglement from the atomic ensemble. In the first step of [28] illustrated in figure 1, an inverted ensemble of atoms creates ASE. This light is entangled with collective degrees of freedom of the atomic ensemble. The excitation in those collective degrees of freedom is then recalled into an output light field with a rephasing $\pi$-pulse. The fact that the quality of entanglement was limited by this recall efficiency is problematic. The low optical depths desirable in the first step to get weak ASE lead to poor recall efficiency. The four-level RASE [31] is slightly better in that separate transitions are used for the ASE and RASE steps. This means that a weak transition can be used for the ASE step and a stronger one for the RASE. Recently [32] further improvement has been suggested by tailoring the spatial density of the ions.

The recall efficiency of the RASE scheme can be improved by placing the atoms inside a low finesse cavity, an approach also used in quantum memories [33, 27, 34]. In this paper we examine this process of cavity enhanced rephased amplified spontaneous emission (CRASE). We will show that in the appropriate regime the CRASE scheme is capable of achieving a recall efficiency of 100%, and in principle perfect multimode, time-separated entanglement.

2. The Hamiltonian for our system

The evolution of CRASE is qualitatively the same as the RASE evolution shown in figure 1. The interaction picture Hamiltonian for our system takes the form (making the rotating wave approximation and setting $\hbar \equiv 1$)

$$H = H_1 + H_2$$ (1)

where

$$H_1 = \sum_{k=1}^{N} \sigma_+^k(t)\sigma_-^k(t) + ig\sum_{k=1}^{N}(\sigma_+^k(t)a(t) - \sigma_-^k(t)a^\dagger(t))$$ (2)

is the usual Jaynes-Cummings Hamiltonian that models the interaction between $N$ two-level atoms and the cavity mode and

$$H_2 = \int_{-\infty}^{\infty} \Delta b^\dagger(\Delta, t)b(\Delta, t)\, d\Delta + i\int_{-\infty}^{\infty} \kappa(\Delta) (b^\dagger(\Delta, t)a(t) - b(\Delta, t)a^\dagger(t))\, d\Delta$$ (3)

models the interaction between the external radiation field and the cavity [35, 36]. The operators $\sigma_+^k(t)$ and $\sigma_-^k(t)$ are raising and lowering operators respectively for atom $k$, $a(t)$ is the destruction operator for the cavity mode, $g$ is the coupling between the atoms and the cavity mode, which we take to be uniform, $b(\Delta, t)$ are destruction operators for
the external radiation modes, $\Delta$ is the detuning from the cavity mode frequency and $\kappa(\Delta)$ is the coupling between the radiation mode with detuning $\Delta$ and the cavity mode.

3. The ASE field

At the beginning of region 1 in figure 1 the atoms are in their excited state. Spontaneous emission by the atoms produces the ASE field. We will assume that the atoms are weakly coupled to the cavity (small $g$) so that the atoms remain predominantly in their excited state throughout region 1. This allows us to approximate each atom as an inverted harmonic oscillator by setting $\sigma_k^+(t) \to s_k(t)$, where $s_k(t)$ are destruction operators satisfying $[s_k(t), s_{k'}^+(t)] = \delta_{kk'}$ [35]. Like the ordinary harmonic oscillator, the eigenstates of an inverted harmonic oscillator form a ladder of equally spaced energy states. The ladder is inverted in the sense that an energy eigenstate $|n\rangle$ contains $n$ units of negative energy so that the state $|n+1\rangle \propto s_k^+ |n\rangle$ has a lower energy than the state $|n\rangle$. We will also approximate the collection of harmonic oscillators by a continuous field by setting $\sqrt{N}s_k(t) \to s(\Delta, t)$. The operators $s(\Delta, t)$ are destruction operators for a collective excitation across oscillators with detuning $\Delta$ and satisfy $[s(\Delta, t), s^+(\Delta', t)] = \delta(\Delta - \Delta')$ in the limit $N \to \infty$. The Hamiltonian $H_1$ then takes the form

$$H_1 = -\int_{-\infty}^{\infty} \Delta s^+(\Delta, t)s(\Delta, t) \, d\Delta + i \int_{-\infty}^{\infty} g(\Delta) \left( s(\Delta, t)a(t) - s^+(\Delta, t)a^+(t) \right) \, d\Delta$$

where $\sqrt{N}g \to g(\Delta)$.

Input-output theory [35] is often used when describing quantum systems interacting with a continuum of radiation modes, such as the situation described by equation (3). The interaction between the excited state atoms and the cavity mode, described by equation (4), is very similar: in both cases the cavity is interacting with a continuum of harmonic oscillators. This allows us to use input-output theory for the atom-cavity interaction also.

Following the standard input-output treatment we assume $\kappa(\Delta)$ and $g(\Delta)$ are slowly varying for our range of frequencies of interest. We can then make the first Markov approximation by setting $\kappa(\Delta) \to \kappa(0) \equiv \sqrt{\gamma_{b,1}/2\pi}$ and $g(\Delta) \to g(0) \equiv \sqrt{\gamma_{a,1}/2\pi}$. In the case of $g$, this is valid when the inhomogeneous broadening of the atoms is much broader than the cavity bandwidth. The loss rate of the bare cavity is $\gamma_{b,1}$. We call $\gamma_{a,1}$ the ‘gain rate’ of the cavity; this is the rate describing the exponential growth of light in the cavity if the cavity mirrors were perfect. We will work in the regime where $\gamma_{a,1} < \gamma_{b,1}$ so that we are below the lasing threshold.

We are now able to obtain an expression for the output ASE field. Solving the Heisenberg equations of motion for $b(\Delta, t)$, $s(\Delta, t)$ and $a(t)$ gives

$$b(\Delta, t) = \exp \left[ -i(\Delta - t_0) \right] b(\Delta, t_0) + \sqrt{\frac{\gamma_{b,1}}{2\pi}} \int_{t_0}^{t} \exp \left[ -i(\Delta - \tau) \right] a(\tau) \, d\tau$$

$$s(\Delta, t) = \exp \left[ i(\Delta - t_0) \right] s(\Delta, t_0) - \sqrt{\frac{\gamma_{a,1}}{2\pi}} \int_{t_0}^{t} \exp \left[ i(\Delta - \tau) \right] a^+(\tau) \, d\tau$$

(5) (6)
Cavity enhanced rephased amplified spontaneous emission

and

\[ a(t) = - \int_{t_0}^{t} \exp \left[ -\frac{\gamma_{b,1} - \gamma_{a,1}}{2} (t - \tau) \right] \left( \sqrt{\gamma_{b,1}} b_{\text{in}}(\tau) + \sqrt{\gamma_{a,1}} s_{\text{in}}^{\dagger}(\tau) \right) \, d\tau \]

\[ + \exp \left[ -\frac{\gamma_{b,1} - \gamma_{a,1}}{2} (t - t_0) \right] a(t_0) \]

where

\[ b_{\text{in}}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp [-i\Delta (t - t_0)] b(\Delta, t_0) \, d\Delta \]

is the radiation field that enters the cavity at time \( t \) and

\[ s_{\text{in}}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp [i\Delta (t - t_0)] s(\Delta, t_0) \, d\Delta \]

is the input atomic field. Time \( t_0 < 0 \) occurs at the beginning of region 1 in figure 1.

Both input fields are vacuum fields with \( \langle b(\Delta, t_0)b(\Delta', t_0) \rangle = \langle s(\Delta, t_0)s(\Delta', t_0) \rangle = 0 \). Note that we name the input atomic field from the point of view of the cavity mode; the input atomic field is the field that drives the cavity mode. Equations (5) and (6) can be used to derive the relations [35]

\[ b_{\text{ASE}}(t) = b_{\text{out}}(t) = b_{\text{in}}(t) + \sqrt{\gamma_{b,1}} a(t) \]

\[ s_{\text{out}}(t) = s_{\text{in}}(t) - \sqrt{\gamma_{a,1}} a^{\dagger}(t) \]

where

\[ b_{\text{out}}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp (-i\Delta t) b(\Delta, 0) \, d\Delta \]

is the ASE field - the radiation field that exits the cavity at time \( t \) - and

\[ s_{\text{out}}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp (i\Delta t) s(\Delta, 0) \, d\Delta \]

is the output atomic field. The output atomic field is a collective de-excitation of atoms, or equivalently an excitation of the inverted harmonic oscillator field.

We will assume that the frequencies of interest of the ASE field is narrowband compared to \( \gamma_{b,1} - \gamma_{a,1} \), the net loss rate of the cavity (see the implementation section for further discussion). We can then adiabatically eliminate the cavity mode, replacing \( b_{\text{in}}(\tau) \) by \( b_{\text{in}}(t) \) and \( s_{\text{in}}(\tau) \) by \( s_{\text{in}}(t) \) in equation (7). Substituting the resulting expression for \( a(t) \) into equations (10) and (11) and letting \( t_0 \to -\infty \) gives the following input-output relations:

\[ b_{\text{ASE}}(t) = \frac{\gamma_{b,1} + \gamma_{a,1}}{\gamma_{b,1} - \gamma_{a,1}} b_{\text{in}}(t) - \frac{2\sqrt{\gamma_{b,1}\gamma_{a,1}}}{\gamma_{b,1} - \gamma_{a,1}} s_{\text{in}}^{\dagger}(t) \]

\[ s_{\text{out}}(t) = \frac{\gamma_{b,1} + \gamma_{a,1}}{\gamma_{b,1} - \gamma_{a,1}} s_{\text{in}}(t) + \frac{2\sqrt{\gamma_{b,1}\gamma_{a,1}}}{\gamma_{b,1} - \gamma_{a,1}} b_{\text{in}}^{\dagger}(t) \]

These relations allow us to determine the output ASE field and output atomic field from the known input radiation and atomic fields.
4. The rephasing \( \pi \)-pulse

The rephasing \( \pi \)-pulse (applied at \( t = 0 \)) induces the following changes in our atoms [37]:

\[
\sigma^k_+(0) \rightarrow -\sigma^k_+(\delta t) \quad (16)
\]

\[
\sigma^k_-(0) \rightarrow \sigma^k_-(\delta t) \quad (17)
\]

where \( \delta t \) is the duration of the \( \pi \)-pulse and \( \sigma^k_\pm(t) = 2\sigma^k_+(t)\sigma^k_-(t) - I \), where \( I \) is the identity operator. The \( \pi \)-pulse inverts the excitation of and rephases the atoms. Like in [28], we model the rephasing pulse as an instantaneous \( \pi \)-pulse and so take \( \delta t \rightarrow 0 \). This is valid if \( (\delta t)^{-1} \) is large compared to the bandwidth of the ASE and RASE fields that we are interested in.

5. The RASE field

After the rephasing \( \pi \)-pulse the atoms are predominantly in their ground state (equation (16)). But the atoms that fell to their ground state in region 1 will be in their excited state and these atoms produce the RASE field. Because the atoms are only weakly excited we approximate the atoms as a field of ordinary harmonic oscillators by setting \( \sigma_-(\Delta, t) \rightarrow d(\Delta, t) \). Here \( d(\Delta, t) \) are destruction operators satisfying \( [d(\Delta, t), d'(\Delta', t)] = \delta(\Delta - \Delta') \).

The interaction picture Hamiltonian for our system takes the form (making the rotating wave approximation and setting \( \hbar \equiv 1 \))

\[
H = \int_{-\infty}^{\infty} \Delta b^\dagger(\Delta, t)b(\Delta, t) \, d\Delta + i\sqrt{\frac{\gamma_{b,2}}{2\pi}} \int_{-\infty}^{\infty} \left( b^\dagger(\Delta, t)a(t) - b(\Delta, t)a^\dagger(t) \right) \, d\Delta
\]

\[
+ \int_{-\infty}^{\infty} \Delta d^\dagger(\Delta, t)d(\Delta, t) \, d\Delta + i\sqrt{\frac{\gamma_{a,2}}{2\pi}} \int_{-\infty}^{\infty} \left( d^\dagger(\Delta, t)a(t) - d(\Delta, t)a^\dagger(t) \right) \, d\Delta \quad (18)
\]

The loss rate of the bare cavity is \( \gamma_{b,1} \) and \( \gamma_{a,1} \) is the rate the atoms would absorb photons from the cavity if the cavity mirrors were perfect. We have allowed for the atom-cavity and radiation-cavity couplings to differ from the couplings for times \( t < 0 \). This allows for the case where either the \( Q \)-factor of the cavity changes or the oscillator strength of the atomic transition used for the ASE and RASE are different [31].

We again make the first Markov approximation and assume that the frequencies of interest of the RASE field is narrowband compared to \( \gamma_{a,2} + \gamma_{b,2} \), the total loss rate of the cavity, allowing us to again adiabatically eliminate the cavity mode. Carrying out analogous calculations to those from the previous section we obtain the following input-output relations:

\[
b_{\text{RASE}}(t) \equiv b_{\text{out}}(t) = \frac{\gamma_{b,2} - \gamma_{a,2}}{\gamma_{b,2} + \gamma_{a,2}} b_{\text{in}}(t) - \frac{2\sqrt{\gamma_{b,2}\gamma_{a,2}}}{\gamma_{b,2} + \gamma_{a,2}} d_{\text{in}}(t) \quad (19)
\]

\[
d_{\text{out}}(t) = \frac{\gamma_{b,2} - \gamma_{a,2}}{\gamma_{b,2} + \gamma_{a,2}} d_{\text{in}}(t) - \frac{2\sqrt{\gamma_{b,2}\gamma_{a,2}}}{\gamma_{b,2} + \gamma_{a,2}} b_{\text{in}}(t) \quad (20)
\]
where
\[ b_{\text{RASE}}(t) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left[ -i\Delta(t - t_1) \right] b(\Delta, t_1) \, d\Delta \]  \hspace{1cm} (21)
is the output RASE field, where \( t_1 \) is any time far in the future,
\[ d_{\text{out}}(t) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left[ -i\Delta(t - t_1) \right] d(\Delta, t_1) \, d\Delta \]  \hspace{1cm} (22)
is the output atomic field,
\[ d_{\text{in}}(t) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp (-i\Delta t) \, d(\Delta, 0) \, d\Delta \]  \hspace{1cm} (23)
is the input atomic field and \( b_{\text{in}}(t) \) is defined by equation (8).

Equation (19) gives the RASE field as a function of the known input radiation field and the currently unknown input atomic field. The input atomic field for region 2 can be related to the output atomic field from region 1 using equation (17) with \( \sigma_{+}^{k}(0) \to s(\Delta, 0) \) and \( \sigma_{-}^{k}(\delta t) \to d(\Delta, \delta t) \). The definitions of \( d_{\text{in}}(t) \) and \( s_{\text{out}}(t) \) (equations (23) and (13)) then give that
\[ d_{\text{in}}(t) = s_{\text{out}}(-t) \]  \hspace{1cm} (24)
in the limit \( \delta t \to 0 \). Equation (24) shows that the input atomic field that drives the cavity mode after the rephasing pulse is equal to the output atomic field that was driven by the cavity mode before the rephasing pulse. Entanglement between \( s_{\text{out}}(t) \) and \( b_{\text{ASE}}(t) \) before the rephasing pulse is translated into entanglement between \( d_{\text{in}}(t) \) and \( b_{\text{ASE}}(-t) \) as a result of equation (24). The field \( b_{\text{RASE}}(t) \) becomes entangled with \( d_{\text{in}}(t) \) and is therefore also entangled with \( b_{\text{ASE}}(-t) \).

Using equations (24) and (15) in equation (19) gives
\[ b_{\text{RASE}}(t) = \frac{\gamma_{b,2} - \gamma_{a,2}}{\gamma_{b,2} + \gamma_{a,2}} b_{\text{in}}(t) - 2\sqrt{\gamma_{b,2} \gamma_{a,2}} \left( \frac{\gamma_{b,1} + \gamma_{a,1}}{\gamma_{b,1} - \gamma_{a,1}} s_{\text{in}}(-t) + \frac{2\sqrt{\gamma_{b,1} \gamma_{a,1}} b_{\text{in}}^{\dagger}(-t)}{\gamma_{b,1} - \gamma_{a,1}} \right) \]  \hspace{1cm} (25)
and this along with equation (14) give the RASE and ASE fields as functions of the input atomic field before the rephasing pulse and input radiation field. The input-output relations describing our system are shown graphically in figure 2.

6. Quantifying entanglement

To concentrate on a single temporal mode for both the ASE and RASE fields we introduce mode operators \( A \) and \( B \) defined by
\[ A \equiv -\int_{-\infty}^{0} g(t)b_{\text{ASE}}(t) \, dt \]  \hspace{1cm} (26)
\[ B \equiv \int_{0}^{\infty} g(t)^{*}b_{\text{RASE}}(t) \, dt \]  \hspace{1cm} (27)
where \( g(t) \) is a temporal mode function satisfying \( g(-t) = g(t) \) and \( \int_{0}^{\infty} |g(t)|^2 \, dt = 1 \). The number of photons in the ASE and RASE modes are then given by \( N_{\text{ASE}} = \langle A^{\dagger}A \rangle \) and \( N_{\text{RASE}} = \langle B^{\dagger}B \rangle \).
Figure 2. Graphical representation of the input-output relations (14) and (25). In the time region 1 the input optical field \( b_{\text{in}}(-t) \) and the input atomic field \( s_{\text{in}}(-t) \) combine like in a non-degenerate parametric amplifier to produce the outputs \( b_{\text{ASE}}(-t) \) and \( s_{\text{out}}(-t) \). The rephasing pulse means that the input atomic field for region 2 \( d_{\text{in}}(t) \) matches the output atomic field from region 1 \( s_{\text{out}}(-t) \). In region 2, the optical and atomic fields interact like on a beamsplitter. For an impedance matched cavity the reflectivity of this beamsplitter is 1. It is well known that in an impedance matched cavity full of atoms any input field gets totally absorbed by the atoms, and none of it escapes as light. What also happens in an impedance matched cavity is that the input atomic field is completely mapped onto the output optical field leading to 100% recall efficiency. This means that if the cavity is impedance matched in region 2, the two output optical fields \( b_{\text{ASE}}(-t) \) and \( b_{\text{RASE}}(t) \) will be maximally entangled.

Equations (14) and (25) give that
\[
A = A_0 \cosh \chi + B_0^\dagger \sinh \chi
\]
\[
B = \sqrt{\epsilon} C_0 - \sqrt{1 - \epsilon} \left( B_0 \cosh \chi + A_0^\dagger \sinh \chi \right)
\]
where \( A_0 \equiv \int_{-\infty}^{0} g(t)b_{\text{in}}(t) \, dt \), \( B_0 \equiv \int_{-\infty}^{0} g(t)^{*}s_{\text{in}}(t) \, dt \), \( C_0 \equiv -\int_{0}^{\infty} g(t)^{*}b_{\text{in}}(t) \, dt \), \( \cosh \chi \equiv (\gamma_{b,1} + \gamma_{a,1})/(\gamma_{b,1} - \gamma_{a,1}) \) and \( \sqrt{\epsilon} \equiv (\gamma_{b,2} - \gamma_{a,2})/(\gamma_{b,2} + \gamma_{a,2}) \). We have that \([c_j, c_k^\dagger] = \delta_{jk}\) and \( \langle c_j^\dagger c_k \rangle = 0\), with \( c_j \in \{ A_0, B_0, C_0 \} \). Equations (28) and (29) take the form of the equations describing the output of a non-degenerate parametric amplifier and beam splitter combination [38], see figure 2.
Equations (28) and (29) give that \( N_{\text{ASE}} = \sinh^2 \chi \) and \( N_{\text{RASE}} = (1 - \epsilon) \sinh^2 \chi \). The recall efficiency of our system is given by \( N_{\text{RASE}}/N_{\text{ASE}} = 1 - \epsilon \). If the cavity is impedance matched in region 2 (\( \gamma_{a,2} = \gamma_{b,2} \)) the recall efficiency is 100%, resulting in perfect entanglement between the ASE and RASE fields.

We will quantify the entanglement between the ASE and RASE fields using the criterion of Duan et al. [39]. For this criterion we introduce the operators

\[
\begin{align*}
u &\equiv \sqrt{\theta} x_A + \sqrt{1 - \theta} x_B \\
u &\equiv \sqrt{\theta} p_A - \sqrt{1 - \theta} p_B
\end{align*}
\]

where \( x_c \equiv (c + c^\dagger)/\sqrt{2} \) and \( p_c \equiv -i(c - c^\dagger)/\sqrt{2} \) are amplitude and phase quadrature fields satisfying \([x_c, p_{c'}] = i\delta_{cc'}\), with \( c, c' \in \{A, B\} \), and \( \theta \) can be any real number in the interval \((0, 1)\). Then a sufficient condition for entanglement between the ASE and RASE photons is that \( \langle \Delta u^2 \rangle + \langle \Delta v^2 \rangle < 1 \) \(\ddagger\). Figure 3 shows a contour plot of \( \langle \Delta u^2 \rangle + \langle \Delta v^2 \rangle \), minimised with respect to \( \theta \), versus \( \sqrt{\epsilon} \) and \( \cosh \chi \). The ASE and RASE fields are entangled for all parameter values considered in this figure.

It can be shown that for our system
\[
\langle \Delta u^2 \rangle + \langle \Delta v^2 \rangle = 1 + 2 \sinh^2 \chi - 2\epsilon(1 - \theta) \sinh^2 \chi - 4\sqrt{\theta - \theta^2} \sqrt{1 - \epsilon} \cosh \chi \sinh \chi \tag{32}
\]

Setting \( \theta = (1 - \epsilon)/(2 - \epsilon) \) gives \( \langle \Delta u^2 \rangle + \langle \Delta v^2 \rangle < 1 \) for all valid values of \( \chi \) and \( \epsilon \). Therefore the ASE and RASE fields are entangled for all valid parameter values.

7. Implementation

To implement our proposal, it is necessary to use two-level atoms with coherence times longer than the combined duration of the ASE and RASE fields. This could be achieved using the long coherence time of rare-earth doped solids [40] or by using Raman transitions in atomic gases [41].

Operation with the cavity impedance matched in region 2 requires either some way of changing the atom oscillator strength, such as the four-level scheme of [31], or a Q-switched cavity because, in order not to cross the lasing threshold, \( \gamma_{b,1} > \gamma_{a,1} \) is required in region 1. In the case of the four-level scheme, the atom-cavity coupling in regions 1 and 2 can be made to differ by a factor \( \gtrsim 10 \) using appropriate rare-earth materials [42, 43]. In the case of Q-switching, it is not satisfactory to simply insert a variable attenuator, as attenuation introduces an unwanted additional output port that reduces the photon recall efficiency. Instead, various methods of active Q-switching can be used that do not introduce unwanted loss: using AOM or EOM components allows for variable control of the output field [44, 45]; the output of a WGM resonator coupled to an optical fiber or prism can be varied by changing the distance between the WGM

\(\ddagger\) The condition given in [39] is that the ASE and RASE photons are entangled if \( \langle \Delta u^2 \rangle + \langle \Delta u^2 \rangle < \chi^2 + 1/\chi^2 \), where \( u \equiv |\lambda|x_A + (1/\lambda)x_B \) and \( v = |\lambda|p_A - (1/\lambda)p_B \) for some real, non-zero \( \lambda \). Setting \( \theta = \lambda^2(\lambda^2 + 1/\lambda^2)^{-1} \) gives the condition used in this paper.
resonator and the fiber or prism [46, 47]; or a thin Fabry-Pérot resonator with variable mirror separation can be used as a mirror with a variable reflectivity [48].

The cavity finesse required for implementation are only moderate, being no larger than that required to achieve impedance matching. This does not pose a great challenge, since the atomic absorption can be made much larger than the limit of current cavity decay rates.

The bandwidth of the output light is determined by either the cavity linewidth or the width of the inhomogeneous broadening. We will only be interested in a band of this light that is narrow compared to \( \min (\gamma_{b,1} - \gamma_{a,1}, \gamma_{b,2} + \gamma_{a,2}) \), since adiabatic elimination of the cavity dynamics is only valid for such a range. Filtering this narrowband signal from the full bandwidth of the output light can be achieved in a number of ways. The easiest to implement is homodyne or heterodyne detection [49]. In this way the filtering problem is reduced to filtering electrical signals. This method is suitable for quantum
repeaters using a continuous variable basis [50, 51]. For quantum repeaters using a discrete Fock basis, filtering using standard techniques such as Fabry-Pérot etalons is a possibility, and when using rare-earth ions one can also use the array of narrow band filtering techniques recently developed based on spectral holeburning [52, 53].

It has been assumed that the rephasing $\pi$-pulse is a perfect $\pi$-pulse. The feasibility of applying a $\pi$-pulse to the whole ensemble is greatly helped by the area theorem [54], which states that a $\pi$-pulse will remain a $\pi$-pulse as it travels through resonant media. This has strong analogues in a cavity [55]. Of course, in practice the $\pi$-pulse won’t be perfect. An imperfect $\pi$ pulse can be considered as a combination of a perfect $\pi$ pulse and a small perturbing pulse. The small perturbing pulse adds unwanted excitation to the atomic field, which adds noise to the RASE field. However, as discussed in [28], this noise disappears shortly after the $\pi$ pulse, since the unwanted excitation of the atomic field will be temporally brief and will rapidly dephase and no longer interact with the cavity mode. The variation in driving strength of the atoms due to variations in the cavity mode field intensity could be removed using hole burning techniques [56, 57].

8. Conclusion

In conclusion, we have analysed rephased amplified spontaneous emission with the atoms placed in an optical cavity. The cavity can alleviate the problem of low recall efficiency, particularly if the cavity is impedance matched when the entangled light is being recalled from the atoms, in which case the recall will theoretically be perfect. Achieving the impedance matched condition during recall requires either a $Q$-switched cavity or some way of switching the atoms’ oscillator strengths, such as in the four-level scheme of Beavan et al. [31], since the ASE field has to be produced below the lasing threshold. We have also shown that entanglement exists between the ASE and RASE fields for all valid parameter values. Theoretically, our system has the potential to achieve time-separated entanglement with perfect recall efficiency, which is indispensable in producing an effective quantum repeater.

Acknowledgements

The authors would like to acknowledge financial support from the Marsden Fund of the Royal Society of New Zealand.

References

Cavity enhanced rephased amplified spontaneous emission

[34] Sabooni M, Li Q, Kr¨ oll S and Rippe L 2013 Phys. Rev. Lett. 110(13) 133604
[38] Walls D and Milburn G 1995 Quantum Optics (Springer)


